Problem 1 4-5

part (a)

Given \( x[n] = 10 \cos (0.2 \pi n - \pi/4) \) and \( f_s = 1000 \text{ Hz} \)

Re:

\[
\omega = \frac{2 \pi f_0 + 2 \pi l}{f_s}; \text{ where } l \text{ is any integer}
\]

In our case,

\[
\omega = 0.2 \pi = \frac{2 \pi f_0 + 2 \pi l}{f_s}, \text{ or } f_0 = 500 (0.2 - 2l)
\]

So when \( l = 0 \), \( f_0 = 500 (0.2) = 100 \text{ Hz} \)

when \( l = 1 \), \( f_0 = 500 (0.2 - 2) = 500 (-1.8) = -900 \text{ Hz} \)

Therefore we could come up with 2 different continuous-time signals \( x_1(t) \) and \( x_2(t) \) whose samples are equal to \( x[n] \). They are

\[
x_1(t) = 10 \cos (2 \pi (100) t - \pi/4)
\]

and

\[
x_2(t) = 10 \cos (2 \pi (-900) t - \pi/4)
\]

\[
= 10 \cos (-(2 \pi (900) t + \pi/4))
\]

Re:

\[
\cos (-\theta) = \cos (\theta)
\]

So

\[
x_2(t) = 10 \cos (- (2 \pi (900) t + \pi/4))
\]

\[
= 10 \cos (2 \pi (900) t + \pi/4)
\]

\[
x_2(t) = 10 \cos (2 \pi (900) t + \pi/4)
\]

part (b)

\[ x[n] \rightarrow \text{Ideal D-to-C Converter} \rightarrow y(t) \]

\( f_s = \frac{1}{f_s} \)
\[ x[n] = 10 \cos (0.2\pi n - \pi/4) \]

Sampling rate = 2000 samples/sec.

Re: The discrete and continuous domains are related by \( \frac{n}{f_s} \leftrightarrow t \) or \( n \leftrightarrow f_s t \)

\[ y(t) = 10 \cos (0.2\pi (2000t) - \pi/4) \]
\[ = 10 \cos (400\pi t - \pi/4) \]

\[ \therefore y(t) = 10 \cos (400\pi t - \pi/4) \]
Problem 2  (4-10)

part (a)

Please see Figure 1 for the plot.

If we consider the sequence (discrete time domain) then

\[ x[n] = \cos (2\pi (13)(0.07n) - \pi/2) \]

Re: The discrete and continuous time domains are related by the following:

\[ \frac{n}{T_s} \leftrightarrow k \quad \text{or} \quad n \leftrightarrow f_s k \]

so

\[ x[n] = \cos (2\pi (0.09)n - \pi/2) = \cos (2\pi (0.09)n - \frac{\pi}{2}) \]

\[ f_{\text{samp}} = \frac{1}{T_s} = \frac{1}{0.07} = 14.28 \text{Hz} \]

In continuous-time, the folded frequency is 14.28 - 13 = 1.28 Hz. Therefore the period is approximately \( \frac{1}{1.28} \approx 0.8 \text{sec} \)

So what we see in the plot is the "folded alias" of the function we wanted to plot.

The period in the plot is approximately 0.8 sec and you see 1.25 periods (\( \frac{1}{0.8} \))

part (b)

Sampling Theorem \( \Rightarrow \) \( f_{\text{samp}} \geq 2F_0 = 2(13) = 26 \)

\[ f_{\text{samp}} = \frac{1}{T_s} \Rightarrow T_s = \frac{1}{f_{\text{samp}}} = \frac{1}{26} \]
To obtain a smooth plot, we need about 20 samples/period, which is a sampling rate of $20F_o$

$$Ts \leq \frac{1}{20F_o} = \frac{1}{20(13)} = \frac{1}{260}$$
part (a)

\[ x[n] = 10 \cos \left( 0.13 \pi n + \frac{\pi}{3} \right) \text{ and } f_s = 1000 \text{ Hz and } T_s = 0.001 \]

Let \( \hat{\omega} = 2\pi f_0 + 2\pi l/k \) where \( l \) is any integer.

In our case,

\[ \hat{\omega} = 0.13 \pi = \frac{2\pi f_0}{f_s} \Rightarrow f_0 = \frac{f_s}{2\pi} \cdot 0.13 \]

when \( l = 0 \)

\[ f_0 = \frac{0.13}{500} = 65 \text{ Hz} \]

when \( l = 1 \)

\[ f_0 = \frac{0.13 - 2}{500} = -935 \text{ Hz} \]

Now we could come up with 2 different continuous time signals \( x_1(t) \) and \( x_2(t) \) whose samples are equal to \( x[n] \). They are the following:

\[ x_1(t) = 10 \cos \left( 2\pi (65) t + \frac{\pi}{3} \right) \]

\[ x_2(t) = 10 \cos \left( 180\pi t + \frac{\pi}{3} \right) \]

\[ x_2(t) = 10 \cos \left( 2\pi (-935) t + \frac{\pi}{3} \right) \]

\[ = 10 \cos \left( -\left( 1870\pi t - \frac{\pi}{3} \right) \right) \]

\[ \text{Re: } \cos(-\theta) = \cos(\theta) \]

\[ x_2(t) = 10 \cos \left( 1870\pi t - \frac{\pi}{3} \right) \]
\[ f_s = 700 \text{ samples/sec} \]
\[ x_1(k) = 26 \cos \left( \frac{2\pi}{7} (200) k + \frac{\pi}{4} \right) + 14 \cos \left( \frac{2\pi}{5} (500) k + \frac{3\pi}{4} \right) \]
\[ x_2(k) \]
\[ n \leftarrow \frac{k}{f_s} \text{ or } k \leftarrow \frac{n}{f_s}. \]
\[ x_1[n] = 26 \cos \left( \frac{2\pi}{7} (200) n + \frac{\pi}{4} \right) + 14 \cos \left( \frac{2\pi}{5} (500) n + \frac{3\pi}{4} \right) \]
\[ x_2[n] \]

At \[ f_0 = 400 \text{ Hz} \], \[ x_1[n] \] does not alias, but \[ x_2[n] \] will alias.

Consider \[ x_1[n] \].
\[ \omega = \frac{4\pi}{7} \left( \frac{f_0}{f_s} + 2\pi k \right) \Rightarrow f_0 = \left[ \frac{4\pi}{7} - 2\pi k \right] \frac{700}{2\pi} \]
when \( k = 0 \):
\[ f_0 = 200 \text{ Hz} \].
\[ \omega \] does not alias.
\[ x_1(k) = 26 \cos \left( \frac{2\pi}{7} (200) k + \frac{\pi}{4} \right) = 26 \cos \left( 400 \frac{\pi}{7} k + \frac{\pi}{4} \right) \]

Consider \[ x_2[n] \].
\[ \hat{\omega} = 2\pi \left( \frac{3}{4} \right) = \frac{2\pi f_0}{f_s} + 2\pi k \Rightarrow f_0 = \left[ \frac{10\pi}{7} - 2\pi k \right] \frac{700}{2\pi} \]
when \( k = 1 \)
\[ f_0 = -200 \text{ Hz} \].

\[ x_2(k) = 14 \cos \left( \frac{2\pi}{7} (-200) k + \frac{3\pi}{4} \right) \]
\[ = 14 \cos \left( -400 \frac{\pi}{7} k - \frac{3\pi}{4} \right) \]
\[ x_2(k) = 14 \cos \left( 400 \frac{\pi}{7} k - \frac{3\pi}{4} \right) \]
\[ \therefore \cos (-\theta) = \cos (\theta) \]

Since \[ x_1(k) \] and \[ x_2(k) \] have the same frequency, they can be combined.
\[ x_1(k) = 26 \cos \left( 400 \frac{\pi}{7} k + \frac{\pi}{4} \right) \]
\[ x_2(k) = 14 \cos \left( 400 \frac{\pi}{7} k - \frac{3\pi}{4} \right) \]
\[ \therefore x_1(k) + x_2(k) = 26 e^{j\frac{\pi}{4}} + 14 e^{-j\frac{3\pi}{4}} = 12 e^{j\frac{\pi}{4}} \]
\[ \therefore x(k) = 12 \cos \left( \frac{2\pi}{7} (200) k + \frac{\pi}{4} \right) = y(k) \]
(a) \[ x(t) = 2 \cos \left( 2\pi (50) t + \frac{\pi}{2} \right) + \cos \left( 2\pi \left( 150 \right) t \right) \]

If \( y(t) = x(t) \), then there was no aliasing that occurred in the conversions and therefore the sampling frequency \( f_s \geq 2f_0 \)

\[ f_s \geq 2 \left( 150 \right) = 300 \]

\[ \therefore f_s \geq 300 \text{ Hz} \]
part (d)

\( \cos(2\pi(150)k) \) folds due to the low sampling frequency.

\( \cos(2\pi(150)k) \) becomes a DC component when

\[ \hat{f} = 2\pi l \]  where \( l \) is an integer.

\[ \hat{f} = \frac{2\pi f_0}{f_s} \]  \therefore if we equate \( \hat{f} \) to \( 2\pi l \) we get

\( \frac{2\pi f_0}{f_s} = 2\pi l \)  \( \Rightarrow \)  \( \frac{f_0}{f_s} = l \)

Take the case where \( l = 1 \),

\[ f_s = f_0 \Rightarrow f_s = 150 \]

We have to check whether the first part 'folds' at that sampling rate.

CHECK:

\[ f_s > 2f_0 \Rightarrow 150 > 2(50) \]  YES

But if \( l = 2 \)

then \( \frac{f_0}{f_s} = 2 \)  \( \Rightarrow \)  \( f_s = 75 \)

CHECK:

\[ 75 \neq 2(50) \]  NO

So the sampling frequency \((f_s)\) we are looking for is

\[ f_s = 150 \text{ Hz} \]
Figure 1