1. The purpose of this exercise is to demonstrate the relationships among the four forms of convergence that we have presented. In each case, \( \omega \) has a uniform probability density function over \([0, 1]\). For each of the following sequences of random variables, determine the pmf (probability mass function) of \( \{Y_n\} \), the senses in which the sequence converges, and the random variable and pmf to which the sequence converges.

(a) 
\[
Y_n(\omega) = \begin{cases} 
1 & \text{if } n \text{ is odd and } \omega < 1/2 \text{ or } n \text{ is even and } \omega > 1/2 \\
0 & \text{otherwise.}
\end{cases}
\]

(b) 
\[
Y_n(\omega) = \begin{cases} 
1 & \text{if } \omega < 1/n \\
0 & \text{otherwise.}
\end{cases}
\]

(c) 
\[
Y_n(\omega) = \begin{cases} 
n & \text{if } \omega < 1/n \\
0 & \text{otherwise.}
\end{cases}
\]

(d) Divide \([0, 1]\) into a sequence of intervals \( \{F_n\} = \{[0, 1], [0, 1/2), [1/2, 1], [0, 1/3], [1/3, 2/3], [2/3, 1], [0, 1/4], ...\} \). Let
\[
Y_n(\omega) = \begin{cases} 
1 & \text{if } \omega \in F_n \\
0 & \text{otherwise.}
\end{cases}
\]

(e) 
\[
Y_n(\omega) = \begin{cases} 
1 & \text{if } \omega < 1/2 + 1/n \\
0 & \text{otherwise.}
\end{cases}
\]

2. Assume that \( x_1, x_2, \ldots, x_L \) are independent of each other and are circularly symmetric complex Gaussian random vectors with mean \( \mu \) and covariance matrix \( Q \). Let 
\[
X = \begin{bmatrix} x_1 & x_2 & \cdots & x_L \end{bmatrix}
\]
Let 
\[
Y = XU,
\]
where \( U \) is an \( L \times L \) unitary matrix. Prove that the columns of \( Y \) are independent of each other.

3. We make \( n \) independent observations \( r_1, \ldots, r_n \), which are real-valued random variables with mean \( m \) and variance \( \sigma^2 \). Let
\[
V = \frac{1}{n} \sum_{j=1}^{n} \left( r_j - \frac{\sum_{i=1}^{n} r_i}{n} \right)^2,
\]
and
\[
S = \frac{\sum_{i=1}^{n} r_i}{n}.
\]
a) Show that \( V \) is a biased estimate of the actual variance, \( \sigma^2 \).

b) If \( r_1, \cdots, r_n \) are real-valued Gaussian random variables, can you show that \( V \) is independent of \( S \), which is the estimate of the mean, \( m \)? If \( r_1, \cdots, r_n \) are not Gaussian, can you show that \( V \) and \( S \) are uncorrelated with each other?

4. Assume that a random variable \( X \) is uniformly distributed in \([0, \theta]\). Let \( x_1, \cdots, x_n \) be \( n \) independent observations of \( X \). Find the maximum likelihood (ML) estimate \( \hat{\theta}_{ML} \) of \( \theta \). Find the bias of \( \hat{\theta}_{ML} \).

5. Justify:

a) if an efficient estimate of the standard deviation \( \sigma \) of a zero-mean Gaussian density exists.

b) if an efficient estimate of the variance \( \sigma^2 \) of a zero-mean Gaussian density exists.

6. The probability density function for a random variable \( Y \) given \( a \) is

\[
f(y|a) = \begin{cases} 
2(a - y) & \text{if } a - 1 \leq y \leq a, \\
0 & \text{otherwise},
\end{cases}
\]

where \( a \) is a deterministic unknown.

a) Find the maximum likelihood (ML) estimate of \( a \) if an observation of \( Y \) is made and the observation is \( y = 2 \).

b) Find the ML estimate of \( a \) if two independent observations of \( Y \) are made and the observations are \( y_1 = 2 \) and \( y_2 = 1.5 \).