Robust Capon Beamforming

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***Abstract*—The Capon beamformer has better resolution and much better interference rejection capability than the standard (data-independent) beamformer, provided that the array steering vector corresponding to the signal of interest (SOI) is accurately known. However, whenever the knowledge of the SOI steering vector is imprecise (as is often the case in practice), the performance of the Capon beamformer may become worse than that of the standard beamformer. We present a natural extension of the Capon beamformer to the case of uncertain steering vectors. The proposed robust Capon beamformer can no longer be expressed in a closed form, but it can be efficiently computed. Its excellent performance is demonstrated via a number of numerical examples.**

***Index Terms*—Adaptive arrays, array errors, robust adaptive beamforming, robust Capon beamforming, signal power estimation.**

I. Introduction and Preliminaries

CONSIDER an array comprising M sensors, and let $\mathbf{R}$ denote the theoretical covariance matrix of the array output vector. We assume that $\mathbf{R}$ has the following form:

$$\mathbf{R} = \sigma_0^2 \mathbf{a}(\theta_0)\mathbf{a}^*(\theta_0) + \sum_{k=1}^{K} \sigma_k^2 \mathbf{a}(\theta_k)\mathbf{a}^*(\theta_k) + \mathbf{Q} \quad (1)$$

where $(\sigma_0^2, \{\sigma_k^2\}_{k=1}^K)$ are the powers of the $(K+1)$ uncorrelated signals impinging on the array; $(\theta_0, \{\theta_k\}_{k=1}^K)$ are the location parameters of the sources emitting those signals [e.g., their directions of arrival (DOAs)]; $(\cdot)^*$ denotes the conjugate transpose; $\mathbf{a}(\cdot)$ is the array steering vector; and $\mathbf{Q}$ is the noise covariance matrix (the "noise" comprises nondirectional signals; hence, $\mathbf{Q}$ usually has full rank as opposed to the other terms in (1) whose rank is equal to one). In what follows, we assume that the first term in (1) corresponds to the signal of interest (SOI) and the remaining rank-one terms to K interferences. Owing to the explicit inclusion of the interference terms in (1), we can assume, without being too restrictive, that the noise covariance matrix is given by $\mathbf{Q} = \sigma^2 \mathbf{I}$. This assumption on $\mathbf{Q}$ is made only for the convenience of the numerical examples, but it has *no* importance for the theoretical development of the robust Capon beamformer (RCB). In the numerical examples (see Section III), we will also make use of the following definition of the steering vector:

$$\mathbf{a}(\theta) = \left[1 \; e^{-j\pi \sin\theta} \; \cdots \; e^{-j\pi(M-1)\sin\theta}\right]^T, \qquad \theta = \text{DOA} \quad (2)$$

which corresponds to a uniform linear array with half-wavelength sensor spacing; in (2), $(\cdot)^T$ denotes the transpose. Similarly to the previous assumption on $\mathbf{Q}$, the form of $\mathbf{a}(\theta)$ above is not used in the theoretical development of the RCB. To end the discussion about (1), we note that the above expression for $\mathbf{R}$ holds for both narrowband signals and wideband signals; in the former case, $\mathbf{R}$ is the covariance matrix at the center frequency; in the latter, $\mathbf{R}$ is the covariance matrix at the center of a given frequency bin.

The *robust beamforming problem* we will deal with in this letter can now be briefly stated as follows: extend the Capon beamformer so as to be able to accurately determine the power of SOI even when only an imprecise knowledge of its steering vector $\mathbf{a}(\theta_0)$ is available. More specifically, we assume that the only knowledge we have about $\mathbf{a}(\theta_0)$ is that it belongs to the following uncertainty ellipsoid:

$$[\mathbf{a}(\theta_0) - \bar{\mathbf{a}}]^* \mathbf{C}^{-1} [\mathbf{a}(\theta_0) - \bar{\mathbf{a}}] \leq 1 \quad (3)$$

where $\bar{\mathbf{a}}$ and $\mathbf{C}$ (a positive definite matrix) are given. A particular instance of (3), which will be considered in the numerical examples, occurs when the array calibration errors are relatively small (and can hence be neglected), but our knowledge of θ_0 is inaccurate, viz. we wrongly assume that the DOA of SOI is $\theta_0 + \Delta$ in lieu of θ_0. In such a case, we typically will choose $\bar{\mathbf{a}} = \mathbf{a}(\theta_0 + \Delta)$; if we also choose $\mathbf{C} = \varepsilon \mathbf{I}$ (for some $\varepsilon > 0$), then (3) becomes

$$\|\mathbf{a}(\theta_0) - \bar{\mathbf{a}}\|^2 \leq \varepsilon \quad \bar{\mathbf{a}} = \mathbf{a}(\theta_0 + \Delta) \quad (4)$$

which is used in [1] and [2] to describe the uncertainty set of $\mathbf{a}(\theta_0)$. We will make use of (4) in the numerical examples (and only there), for the sake of simplicity. Of course, in any application in which we have enough informationrmation to choose $\mathbf{C} \not\sim \mathbf{I}$ so as to describe the shape of the uncertainty set of $\mathbf{a}(\theta_0)$ as well as possible, we will use (3) in lieu of (4). To continue this remark on (3) and (4), we note that in some cases the steering vector may be known to lie in an ellipsoid that is effectively flat in one or more dimensions (see [3] and [4]). In such cases, the use of (3) may lead to a numerically ill-conditioned problem and is not recommended. The modification of (3) to include the flat ellipsoid case is easy [3], [4]. The modification of the RCB approach in this letter to accommodate such a case is a bit more complicated and will be presented elsewhere (see [5][1]).

There is significant literature on robust beamforming, and recent critical reviews can be found in [1]–[4] and [6]. As explained in the cited references, most of the early suggested methods are rather ad hoc. Only recently have some methods with a clear theoretical background been proposed (e.g., see [1]–[4], which make explicit use of an uncertainty set [such as (4)], unlike the early methods). The RCB proposed in this letter also has a firm theoretical basis. More specifically, we couple

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[1]See also http://www.sal.ufl.edu/wang/RCBDiagLoad.pdf

the formulation of standard (nonrobust) Capon beamformer (SCB) in [7] with the uncertainty set in (3) and show that the so-obtained robust beamforming problem can be efficiently solved (see Section II for details). Herein, we focus on the SOI power estimation problem, but the proposed RCB can also be used for DOA as well as signal waveform estimation.

In a separate full paper [5], a detailed comparison with the RCB approaches recently proposed in [1]–[4] will be provided. As briefly explained in Section II, the approaches in the cited papers and the one in this letter are based on rather different formulations of the robust beamforming problem, and hence their comparison is not a straightforward task.

II. Robust Capon Beamforming

The common formulation of the beamforming problem that leads to the SCB is as follows (e.g., see [8] and [9]).

a) Determine the $M \times 1$ vector $\mathbf{w}_0$ that is the solution to the following linearly constrained quadratic problem:

$$\min_{\mathbf{w}} \mathbf{w}^* \mathbf{R} \mathbf{w} \quad \text{subject to} \quad \mathbf{w}^* \mathbf{a}(\theta_0) = 1 \tag{5}$$

(in applications, $\mathbf{R}$ is replaced by the sample covariance matrix).

b) Use $\mathbf{w}_0^* \mathbf{R} \mathbf{w}_0$ as an estimate of σ_0^2. The solution to (5) is easily derived (assuming that the inverse of $\mathbf{R}$ exists)

$$\mathbf{w}_0 = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta_0)}{\mathbf{a}^*(\theta_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0)}. \tag{6}$$

Using (6) in Step b) above yields the following estimate of σ_0^2:

$$\tilde{\sigma}_0^2 = \frac{1}{\mathbf{a}^*(\theta_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0)}. \tag{7}$$

The recent RCB approaches in [1]–[4] extended Step a) above to take into account the fact that when there is uncertainty in $\mathbf{a}(\theta_0)$, the constraint on $\mathbf{w}^*\mathbf{a}(\theta_0)$ in (5) should be replaced with a constraint on $\mathbf{w}^*\mathbf{a}$ for any vector $\mathbf{a}$ in the uncertainty set. Then, the so-obtained $\mathbf{w}$ is used in $\mathbf{w}^*\mathbf{R}\mathbf{w}$ to derive an estimate of σ_0^2, as in Step b) of SCB.

Our approach is different. We use the reformulation of the Capon beamforming problem in [7], which we present below in a simple form, to which we append the uncertainty set in (3). Proceeding in this way, we *directly* obtain a robust estimate of σ_0^2, without any intermediate calculation of the vector $\mathbf{w}$. To describe the details of our approach, we first prove that $\tilde{\sigma}_0^2$ in (7) is the solution to the following problem (also see [7]):

$$\max_{\sigma^2} \sigma^2 \quad \text{subject to} \quad \mathbf{R} - \sigma^2 \mathbf{a}(\theta_0)\mathbf{a}^*(\theta_0) \geq 0 \tag{8}$$

where the notation $\mathbf{A} \geq 0$ (for any Hermitian matrix $\mathbf{A}$) means that $\mathbf{A}$ is positive semidefinite. The previous claim follows from the next readily verified equivalences (here $\mathbf{R}^{-1/2}$ is the Hermitian square root of $\mathbf{R}^{-1}$)

$$\begin{aligned} &\mathbf{R} - \sigma^2 \mathbf{a}(\theta_0)\mathbf{a}^*(\theta_0) \geq 0 \Leftrightarrow \\ &\mathbf{I} - \sigma^2 \mathbf{R}^{\frac{-1}{2}} \mathbf{a}(\theta_0)\mathbf{a}^*(\theta_0)\mathbf{R}^{\frac{-1}{2}} \geq 0 \Leftrightarrow \\ &1 - \sigma^2 \mathbf{a}^*(\theta_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0) \geq 0 \Leftrightarrow \\ &\sigma^2 \leq \frac{1}{\mathbf{a}^*(\theta_0)\mathbf{R}^{-1}\mathbf{a}(\theta_0)} = \tilde{\sigma}_0^2. \end{aligned} \tag{9}$$

Hence $\sigma^2 = \tilde{\sigma}_0^2$ is indeed the largest value of σ^2 for which the constraint in (8) is satisfied. Note that (8) can be interpreted as a *covariance fitting problem*: given $\mathbf{R}$ and $\mathbf{a}(\theta_0)$, we wish to determine the largest possible SOI term $\sigma^2\mathbf{a}(\theta_0)\mathbf{a}^*(\theta_0)$ that can be a part of $\mathbf{R}$ under the natural constraint that the residual covariance matrix be positive semidefinite. When $\mathbf{a}(\theta_0)$ is uncertain, so that we only know that it belongs to the set (3), a direct extension of the previous covariance fitting interpretation leads to the following robustified problem for estimating σ_0^2:

$$\begin{aligned} \max_{\sigma^2, \mathbf{a}} \sigma^2 \quad & \text{subject to } \mathbf{R} - \sigma^2 \mathbf{a}\mathbf{a}^* \geq 0 \\ & \text{for any } \mathbf{a} \text{ satisfying } (\mathbf{a}-\bar{\mathbf{a}})^*\mathbf{C}^{-1}(\mathbf{a}-\bar{\mathbf{a}}) \leq 1 \end{aligned} \tag{10}$$

(where $\bar{\mathbf{a}}$ and $\mathbf{C}$ are given).

Observe that both the power and the steering vector of SOI are treated as unknowns in the above problem and, hence, that there is a "scaling ambiguity" in the SOI covariance term in the sense that $(\sigma^2, \mathbf{a})$ and $(\sigma^2/\alpha, \alpha^{1/2}\mathbf{a})$ (for any $\alpha > 0$) give the same term $\sigma^2\mathbf{a}\mathbf{a}^*$. This observation is important, as it implies that $\hat{\sigma}_0^2$ obtained from (10) may easily be an *overestimate* of σ_0^2. To see why this is so, think of the fact that, for example, the pair $[\sigma_0^2/\alpha,\ \alpha^{1/2}\mathbf{a}(\theta_0)]$ (with $\alpha < 1$ and such that $\alpha^{1/2}\mathbf{a}(\theta_0)$ belong to the uncertainty set) will be preferred by the criterion in (10) (i.e., $\max \sigma^2$) to the true pair $[\sigma_0^2,\ \mathbf{a}(\theta_0)]$. Fortunately, this problem is easily overcome in our framework. To explain how this can be done, let $(\hat{\sigma}_0^2, \hat{\mathbf{a}}_0)$ be the solution to (10). Because $\|\mathbf{a}(\theta_0)\|^2 = M$ [e.g., see (2)] we can simply eliminate the aforementioned "scaling ambiguity" by replacing $\hat{\mathbf{a}}_0$ with $\hat{\hat{\mathbf{a}}}_0 = M^{1/2}\hat{\mathbf{a}}_0/\|\hat{\mathbf{a}}_0\|$ [so that $\|\hat{\hat{\mathbf{a}}}_0\|^2 = M$ as for the true steering vector $\mathbf{a}(\theta_0)$] and accordingly $\hat{\sigma}_0^2$ by $\hat{\hat{\sigma}}_0^2 = \hat{\sigma}_0^2\|\hat{\mathbf{a}}_0\|^2/M$ (so that $\hat{\sigma}_0^2\hat{\mathbf{a}}_0\hat{\mathbf{a}}_0^* = \hat{\hat{\sigma}}_0^2\hat{\hat{\mathbf{a}}}_0\hat{\hat{\mathbf{a}}}_0^*$). Hence, we propose to estimate σ_0^2 as

$$\hat{\hat{\sigma}}_0^2 = \frac{\hat{\sigma}_0^2\|\hat{\mathbf{a}}_0\|^2}{M}. \tag{11}$$

The numerical examples in the next section confirm that $\hat{\hat{\sigma}}_0^2$ is a (much) more accurate estimate of σ_0^2 than $\hat{\sigma}_0^2$.

Remark: The approaches of [1]–[4] do not provide any direct estimate $\hat{\mathbf{a}}_0$, unlike our approach. Hence, they do not dispose of a simple way [such as (11)] to correct the overestimation of the SOI power that is likely a problem for all robust beamforming approaches (this problem was in fact ignored in [1]–[4]). Note that SOI power estimation is the main goal in many applications including radar, sonar, and acoustic imaging.

The RCB problem (10) can be readily reformulated as a *semidefinite program*. Indeed, using a new variable $\rho = 1/\sigma^2$ along with the standard technique of Schur complements (e.g., see [9] and [10]) we can rewrite (10) as

$$\begin{aligned} \min_{\rho, \mathbf{a}} \rho \quad \text{subject to} \quad & \begin{bmatrix} \mathbf{R} & \mathbf{a} \\ \mathbf{a}^* & \rho \end{bmatrix} \geq 0 \\ & \begin{bmatrix} \mathbf{C} & \mathbf{a}-\bar{\mathbf{a}} \\ (\mathbf{a}-\bar{\mathbf{a}})^* & 1 \end{bmatrix} \geq 0. \end{aligned} \tag{12}$$

The constraints in (12) are so-called linear matrix inequalities; hence, (12) is a semidefinite program that can be solved in a time that is a polynomial function of M (e.g., see [10] and also below). In the numerical examples in Section III, we use the SeDuMi software (see [11]) to solve (12). SeDuMi is general software that can be directly applied to (12) but which does not

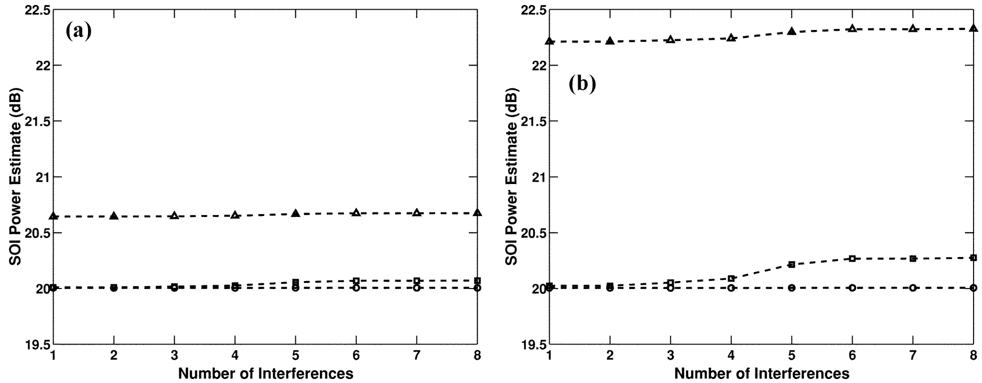

Fig. 1. SOI power estimates $\tilde{\sigma}_0^2$ (bottom dashed line), $\hat{\sigma}_0^2$ (top dashed line), and $\hat{\hat{\sigma}}_0^2$ (middle dashed line), versus K for (a) $\varepsilon = 0.05$ and (b) $\varepsilon = 0.5$. The true SOI power is 20 dB, and $\varepsilon_0 = 0$ (i.e., no mismatch).

exploit the particular structure of (12) and requires in the order of $O(M^6)$ floating-point operations (flops). While this is quite a bit more than the $O(M^3)$ flops required by SCB, it is definitely manageable. Furthermore, the structure of (12) can be exploited to solve this problem in a time that is comparable with that required by SCB. Such a possibility along with other computational issues will be investigated in [5]. Here we only provide a brief account of the main idea.

For any given $\mathbf{a}$, the solution ρ_0 to (12) is given by [see (9)] $\rho_0 = \mathbf{a}^*\mathbf{R}^{-1}\mathbf{a}$. Hence, (12) can be reduced to the following problem:

$$\min_{\mathbf{a}} \mathbf{a}^*\mathbf{R}^{-1}\mathbf{a} \quad \text{subject to } (\mathbf{a}-\bar{\mathbf{a}})^*\mathbf{C}^{-1}(\mathbf{a}-\bar{\mathbf{a}}) \leq 1. \quad (13)$$

Evidently, the solution to (13) will occur on the boundary of the constraint set (under the natural condition that the trivial steering vector $\mathbf{a} = \mathbf{0}$ does not belong to the set in (13)), and therefore we can reformulate (13) as the following quadratic problem with a quadratic equality constraint:

$$\min_{\mathbf{a}} \mathbf{a}^*\mathbf{R}^{-1}\mathbf{a} \quad \text{subject to } (\mathbf{a}-\bar{\mathbf{a}})^*\mathbf{C}^{-1}(\mathbf{a}-\bar{\mathbf{a}}) = 1. \quad (14)$$

This problem can be solved in $O(M^3)$ flops by using the *Lagrange multiplier methodology* (see [5]; also see [3] and [4] for a related approach). It can also be solved slightly less efficiently by using the second-order cone programming approach (e.g., see [12] for a general discussion on second-order cone programs). In conclusion of the discussion on this aspect, note that the semidefinite programming formulation of RCB is useful for gaining a *theoretical understanding* of the RCB approach but it is less suitable than other formulations [such as (14)] from a practical implementation viewpoint.

III. Numerical Examples

Our main motivation for studying the RCB problem was an acoustic imaging application where the goal was to estimate the SOI power in the presence of strong interferences as well as some uncertainty in the SOI DOA. Consequently, in this section we consider scenarios with several strong interferences, in which the estimation of the SOI power is particularly challenging. We assume a spatially white noise whose covariance matrix is given by $\mathbf{Q} = \mathbf{I}$. The power of SOI is $\sigma_0^2 = 20$ dB, and the interference powers are $\sigma_1^2 = \cdots = \sigma_K^2 = 40$ dB. The SOI and interference directions of arrival are $\theta_0 = 10^\circ$, $\theta_1 = -75^\circ$, $\theta_2 = -60^\circ$, $\theta_3 = -45^\circ$, $\theta_4 = -30^\circ$, $\theta_5 = -10^\circ$, $\theta_6 = 25^\circ$, $\theta_7 = 35^\circ$, and $\theta_8 = 50^\circ$. (as it will be explained shortly, we consider $K \leq 8$). We assume a uniform linear array with $M = 10$ sensors for which the steering vector is given by (2). We also assume that the theoretical covariance matrix $\mathbf{R}$ is available (sampling effects due to the use of the sample covariance matrix in lieu of $\mathbf{R}$ are minor for as few as 15 snapshots (see [5]), and to simplify the present discussion we do not consider them herein).

In all the examples, we use $\bar{\mathbf{a}} = \mathbf{a}(\theta_0 + \Delta)$ and $\mathbf{C} = \varepsilon\mathbf{I}$ [as in (4)]. To show empirically that the choice of ε is not a critical issue for our RCB approach, in each case considered below we will present numerical results for several values of ε. We use the notation ε_0 to denote the minimum value of ε for which $\mathbf{a}(\theta_0)$ belongs to the set in (4), and we let ε denote any other choice of this user parameter. The aforementioned insensitivity of the performance of our approach to the choice of ε (in a "reasonable" interval around ε_0) is evidently a desirable feature that we will illustrate numerically but will not attempt to explain theoretically here.

We will vary the number of interferences from $K = 1$ to $K = 8$. For more than $K = 8$ interferences, we observed a certain performance degradation of the RCB, which was somewhat expected. Indeed, for an array with $M = 10$ sensors, the SCB can cancel generally up to $M - 1 = 9$ interferences. This is so, since the spatial filter $\mathbf{w}$ of SCB needs one degree of freedom (DOF) to satisfy the constraint in (5) and one DOF for nulling each interference: as $\mathbf{w}$ has M DOFs, it follows that generally at most $K = M - 1$ interferences can be dealt with by SCB. For the RCB, the (implicit) requirement to pass the SOI with an uncertain steering vector is more demanding, and hence the number of interferences that can be dealt with is smaller than nine.

In Figs. 1 and 2, we show both $\hat{\sigma}_0^2$ and $\hat{\hat{\sigma}}_0^2$ to confirm that the correction in (11) was in effect necessary. In Fig. 1, we show $\tilde{\sigma}_0^2$, $\hat{\sigma}_0^2$, and $\hat{\hat{\sigma}}_0^2$, versus K, for the no-mismatch case; hence $\Delta = 0$ in (4) and consequently $\varepsilon_0 = 0$. As can be seen from Fig. 1, the performance degradation of $\hat{\hat{\sigma}}_0^2$ compared with $\tilde{\sigma}_0^2$, in this *ideal* situation for the SCB, is minor; moreover, this degradation increases only slightly with increasing ε. We also see from Fig. 1 (and Fig. 2) that $\hat{\sigma}_0^2$ significantly overestimates σ_0^2 as predicted.

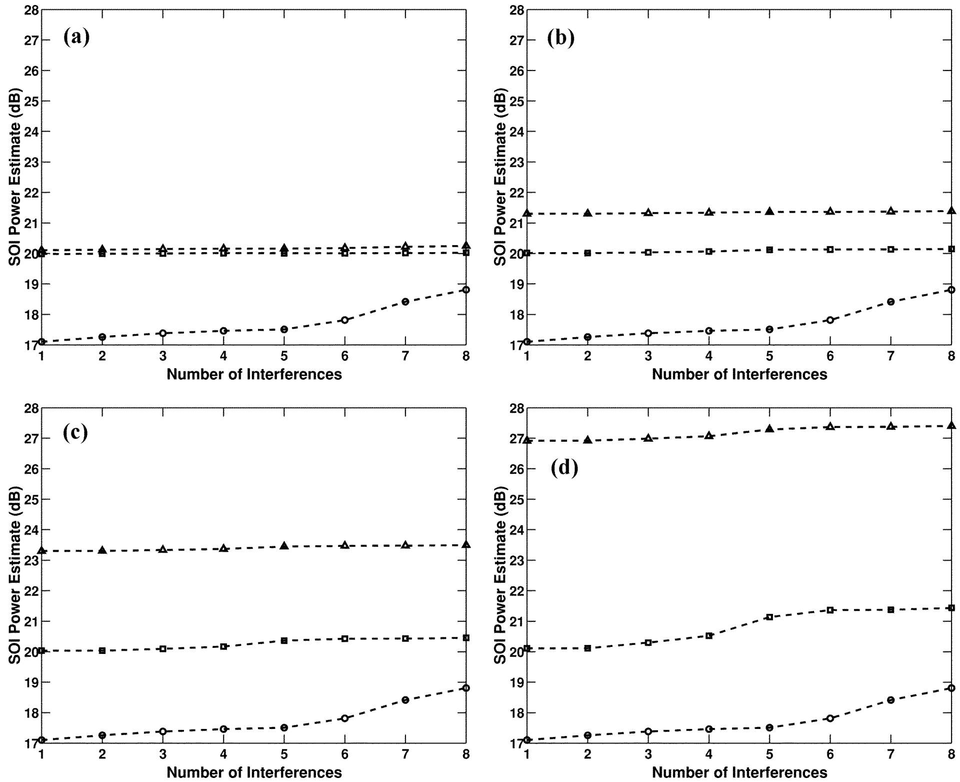

Fig. 2. SOI power estimates $\tilde{\sigma}_0^2$ (bottom dashed line), $\hat{\sigma}_0^2$ (top dashed line), and $\hat{\hat{\sigma}}_0^2$ (middle dashed line), versus K for (a) $\varepsilon = 0.01$, (b) $\varepsilon = 0.2$, (c) $\varepsilon = 1$, and (d) $\varepsilon = 3$. The true SOI power is 20 dB, and $\varepsilon_0 = 0.0332$ (corresponding to $\Delta = 0.2°$).

Fig. 2 shows the same type of curves as Fig. 1, but for a mismatched case in which $\Delta = 0.2°$, and accordingly $\varepsilon_0 = 0.0332$. As can be seen, even a relatively small Δ can cause a significant degradation of the SCB performance. On the other hand, the performance of $\hat{\hat{\sigma}}_0^2$ obtained via our RCB approach is quite good for a wide range of values of ε. Perhaps somewhat surprising at first sight, the performance of SCB improves as K increases. However, there is a simple explanation for this kind of behavior of SCB: for a small value of K, the SCB has enough many DOFs to cancel not only the interference(s) but also the SOI (which comes from another DOA, θ_0, than the assumed one ($\theta_0 + \Delta$) and hence is treated as an interference by SCB). This results in a significant underestimation of σ_0^2. On the other hand, as K increases, the SCB focuses on canceling the interferences (which are much stronger than the SOI) and hence pays less attention to the cancelation of SOI.

IV. Conclusion

Beamforming is a ubiquitous task in array signal processing with applications, among others, in radar, sonar, acoustics, astronomy, communications, and medical imaging. The theoretical advantages of the data-dependent beamforming approaches, such as the SCB, over the data-independent beamformer are easily lost when the knowledge of the array steering vector is imprecise (as is often the case in practice). The RCB approach introduced in this letter to cure this problem of SCB has a natural and firm theoretical foundation, as well as a much better performance than the SCB at a comparable computational cost. Hence, the RCB may help restore the appeal of the data- dependent beamforming in applications with uncertain steering vectors.

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