EEL 6537

Homework #1 – ML Parameter Estimation and CRB Due 2/5/08

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1. Consider the data set

$$\mathbf{x}_l = \beta \mathbf{a} s_l + \mathbf{e}_l, \quad l = 1, 2, \cdots, L,$$

where $\mathbf{x}_l$ denote the lth $M \times 1$ data vector, $\mathbf{a} = [\ 1 \quad 1 \quad \cdots \quad 1\ ]^T$ (an $M \times 1$ vector) with $()^T$ denoting the transpose, β is a complex-valued unknown scalar, $\{s_l\}$ is the known signal waveform, and $\mathbf{e}_l$ denotes the lth error vector. The error vectors $\mathbf{e}_l$, $l = 1, 2, \cdots, L$, are independently and identically distributed zero-mean circularly symmetric complex Gaussian random vectors with an unknown and arbitrary covariance matrix $\mathbf{Q}$. The problem of interest herein is to determine the maximum likelihood (ML) estimate of β from $\{\mathbf{x}_l\}_{l=1}^{L}$ and its Cramer-Rao bound (CRB).

a) Determine the log-likelihood function (logarithm of the pdf) l_1 of $\{\mathbf{x}_l\}_{l=1}^{L}$.

b) Derive the ML estimate of β. (Consider first setting $\partial l_1 / \partial \mathbf{Q}_{ij} = 0$, where $\mathbf{Q}_{ij}$ denotes the ijth element of $\mathbf{Q}$.)

c) Calculate the CRB of β.

d) Perform the following numerical simulations. Let $M = 5$, $L = 50$, $\{s_l\}$ be a sequence consisting of 1's and 0's with equal probability (fixed for all Monte-Carlo trials), and

$$\mathbf{Q} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}.$$

Generate 100 realizations of the data vectors using different error realizations for each mean-squared error (MSE) evaluation of the ML estimate of β. Consider $\rho = 0, 0.5, 0.9$. For each case, generate the MSE of the ML estimate of β as a function of σ^2 and compare with the corresponding CRB. Comment on the effect of ρ.

e) Write a report that details your findings. Be concise and complete.

2. Consider the data set $\mathbf{y}_l$, $l = 1, \cdots, L$, where $\{\mathbf{y}_l\}$ are i.i.d. zero-mean circularly symmetric complex Gaussian random vectors with covariance $\alpha \mathbf{Q}_0 + \beta \mathbf{Q}_1$, where $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are known, $\alpha \geq 0$ and $\beta \geq 0$ are unknown real-valued non-negative scalars.

a) Determine the log-likelihood function (logarithm of the pdf) of $\{\mathbf{y}_l\}_{l=1}^{L}$.

b) Derive the ML estimates of α and β.

c) Calculate the CRB of α and β.

d) Perform some numerical simulations comparing the MSE of the ML estimates of α and β with the corresponding CRB.

e) Write a report that details your findings and insights. Be concise and complete.