EEL 6537

Homework #2 – Convergence, Independence, ... Due 2/28/06

1. The purpose of this exercise is to demonstrate the relationships among the four forms of convergence that we have presented. In each case, ω has a uniform probability density function over $[0,1]$. For each of the following sequences of random variables, determine the pmf (probability mass function) of $\{\mathbf{Y}_n\}$, the senses in which the sequence converges, and the random variable and pmf to which the sequence converges.

(a)

$$Y_n(\omega) = \begin{cases} 1 & \text{if } n \text{ is odd and } \omega < 1/2 \text{ or } n \text{ is even and } \omega > 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$Y_n(\omega) = \begin{cases} n & \text{if } \omega < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

(d) Divide [0,1] into a sequence of intervals $\{\mathbf{F}_n\} = \{[0,1], [0,1/2), [1/2,1], [0,1/3], [1/3,2/3], [2/3,1], [0,1/4], ...\}$. Let

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega \in F_n \\ 0 & \text{otherwise.} \end{cases}$$

(e)

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega < 1/2 + 1/n \\ 0 & \text{otherwise.} \end{cases}$$

2. Assume that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_L$ are independent of each other and are circularly symmetric complex Gaussian random vectors with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{Q}$. Let $\mathbf{X} = [\ \mathbf{x}_1 \ \ \mathbf{x}_2 \ \ \cdots \ \ \mathbf{x}_L\ ]$. Let

$$\mathbf{Y} = \mathbf{X}\mathbf{U},$$

where $\mathbf{U}$ is an $L \times L$ unitary matrix. Prove that the columns of $\mathbf{Y}$ are independent of each other.

3. We make n independent observations $r_1, \cdots, r_n$, which are real-valued random variables with mean m and variance σ^2. Let

$$V = \frac{1}{n}\sum_{j=1}^{n}\left(r_j - \sum_{i=1}^{n}\frac{r_i}{n}\right)^2,$$

and

$$S = \sum_{i=1}^{n}\frac{r_i}{n}.$$

a) Show that V is a biased estimate of the actual variance, σ^2.

b) If $r_1, \cdots, r_n$ are real-valued Gaussian random variables, can you show that V is independent of S, which is the estimate of the mean, m? If $r_1, \cdots, r_n$ are not Gaussian, can you show that V and S are uncorrelated with each other?

4. Assume that a random variable X is uniformly distributed in $[0, \theta]$. Let $x_1, \cdots, x_n$ be n independent observations of X. Find the maximum likelihood (ML) estimate $\hat{\theta}_{\text{ML}}$ of θ. Find the bias of $\hat{\theta}_{\text{ML}}$.

5. Justify:

a) if an efficient estimate of the standard deviation σ of a zero-mean Gaussian density exists.

b) if an efficient estimate of the variance σ^2 of a zero-mean Gaussian density exists.

6. The probability density function for a random variable Y given a is

$$f(y|a) = \begin{cases} 2(a-y) & \text{if } a-1 \leq y \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

where a is a deterministic unknown.

a) Find the maximum likelihood (ML) estimate of a if an observation of Y is made and the observation is $y = 2$.

b) Find the ML estimate of a if two independent observations of Y are made and the observations are $y_1 = 2$ and $y_2 = 1.5$.