EEL 6537, Homework #3 **Due 4/17/08**

1. If we wish to model the true PSD

$$P_{xx}(f) = \begin{cases} 2 & \text{for } |f| \le 0.25, \\ 0 & \text{for } 0.25 < |f| \le 0.5. \end{cases}$$

by the Gaussian PSD (see the course slides on the web), then $r_{xx}[0]$ and σ_f must be estimated. Assume that enough data are available so that the ACF estimates $\hat{r}_{xx}[0]$, $\hat{r}_{xx}[1]$ are equal to the true ACF samples. Find the estimates of the unknown parameters by letting

$$r_{xx}[0] = \hat{r}_{xx}[0]$$
$$r_{xx}[1] = \hat{r}_{xx}[1]$$

where $r_{xx}[0]$, $r_{xx}[1]$ are the ACF samples corresponding to the Gaussian PSD model. Plot the estimated PSD.

Hint:

(a) To make the problem easier, let

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P(f) e^{j2\pi fn} df \approx \int_{-\infty}^{\infty} P(f) e^{j2\pi fn} df$$

(b) Plot both true PSD and estimated Gaussian PSD on the same plot. Use MATLAB.

(c) Discuss the method accuracy when used to estimate $P(f)$ that satisfies the model. How do the parameters in the model affect the approximation in (a) above?

2. Prove that a necessary and sufficient condition for $\mathcal{A}(z) = 1 + a[1]z^{-1} + a[2]z^{-2}$ to be minimum-phase is that $|k_1| < 1$, $|k_2| < 1$. Restrict the proof to the case of real $a[1]$, $a[2]$ coefficients.

Hint:

(a) The roots may be both real or both complex. Assume $x(n)$ is real.

(b) The k_1 and k_2 are defined in Levinson-Durbin algorithm. Show that k_1 and k_2 are real.

(c) Show that

$$1 + a[1]z^{-1} + a[2]z^{-2} = 1 + k_1 z^{-1} + k_2 z^{-2}(k_1 z + 1)$$

(d) Let α be a root of $1 + a[1]z^{-1} + a[2]z^{-2} = 0$. Show that

$$k_2 = -\alpha \frac{\alpha + k_1}{k_1 \alpha + 1} \triangleq -\alpha q$$

(e) Show that if $|k_1| < 1$,

$$|q|^2 = \begin{cases} \geq 1, & \text{for } |\alpha| \geq 1, \\ < 1, & \text{for } |\alpha| < 1. \end{cases}$$

3. Use straightforward and LDA methods to solve

$$\begin{bmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.9 \\ 0.7 & 0.9 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \end{bmatrix}$$

4. Consider the problem of fitting the data $\{x[0], x[1], \ldots, x[N-1]\}$ by the sum of a DC signal and a sinusoid as

$$\hat{x}[n] = \mu + A_c \exp{(j2\pi f_0 n)}, \quad n = 0, 1, \ldots, N-1$$

The complex DC level μ and the complex sinusoid amplitude A_c are unknown. We may view the determination of μ, A_c as the solution of the over-determined set of linear equations

$$\begin{bmatrix} 1 & 1 \\ 1 & \exp{(j2\pi f_0)} \\ \vdots & \vdots \\ 1 & \exp{(j2\pi f_0[N-1])} \end{bmatrix} \begin{bmatrix} \mu \\ A_c \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Find the least squares solution for μ and A_c. Now assume that $f_0 = k/N$, where k is a nonzero integer in the range $[-N/2, N/2-1]$ for N even and $[-(N-1)/2, (N-1)/2]$ for N odd and again find the least squares solution.

5. For AR signals whose poles are near the unit circle, it is also better to use $\hat{r}(k)$ with large lags. Derive over-determined Yule-Walker equations for AR(p) signals.

6. Prove that for an AR(p) signal

$$\delta_p = \delta_{p+1} = \delta_{p+2} = \cdots$$

This result also means that for an AR(p) signal, the linear prediction error remains constant when the linear prediction order is greater than or equal to p.

7. Consider an AR(3) signal. Show that $r(0)$, $r(1)$, $r(2)$, $r(3)$ are functions of a_1, a_2, a_3 and σ^2.

8. Show that the exact likelihood function for a real Gaussian AR(p) process is

$$p(\mathbf{x}; \mathbf{a}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2} det^{1/2}(\bar{\mathbf{R}}_{xx})} \exp[-\frac{1}{2\sigma^2} S(\mathbf{a})]$$

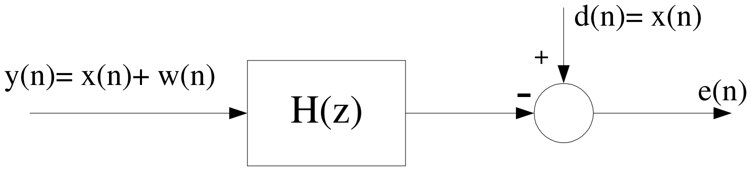

Figure 1: Diagram of filtering in Problem 9.

where

$$S(\mathbf{a}) = \sum_{n=p}^{N-1} (\sum_{k=0}^{p} a[k]x[n-k])^2 + \mathbf{x}_0^T \bar{\mathbf{R}}_{xx}^{-1} \mathbf{x}_0$$

and $\mathbf{x}_0 = [x[0]\ x[1]\ \cdots\ x[p-1]]^T$. The autocorrelation matrix $\bar{\mathbf{R}}_{xx}$ is the usual autocorrelation matrix $\mathbf{R}_{xx}$ divided by σ^2. Note that $\bar{\mathbf{R}}_{xx}$ depends only on the AR filter parameters.

Hint:

(a) See Notes and Problem 7 for better understanding.

(b) The $p(\mathbf{x}; \mathbf{a}, \sigma^2)$ is PDF of $\mathbf{x}$ given $\mathbf{a}$ and σ^2. This is equivalent to our notes notation $f(x[0], \ldots, x[N-1]|\ a_1, \ldots, a_p, \sigma^2)$.

9. Consider the following filtering problem in Figure 1.

Let $\{x(n)\}$ be zero mean, $r_{xx}(k) = \alpha^{|k|}$. Let $\{w(n)\}$ be zero mean, white, i.e., $r_{ww}(k) = \rho\delta(k)$. Assume $\{x(n)\}$ and $\{w(n)\}$ are uncorrelated, we wish to find optimal $H(z)$ to predict $x(n)$. Let $\alpha = 0.8$, $\rho = 1$:

(a) Find optimal noncausal $h(k)$.

(b) Find optimal causal $h(k)$.

(c) Let $H(z) = \sum_{k=-L}^{L} h(k)z^{-k}$, find an expression for the optimal $h(k)$. Let $L = 1$, find $h(-1)$, $h(0)$, $h(1)$.