EEL 6537 Spectral Estimation Project #1

Wenxing Ye 4949-3688

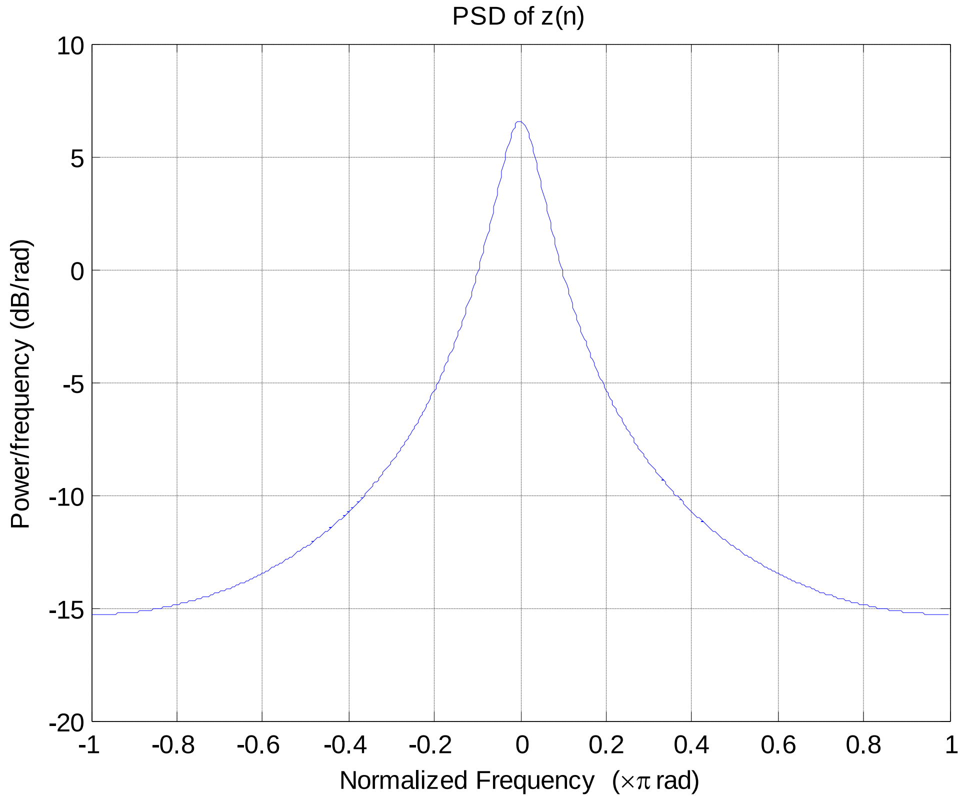
1)

a) $r_{uu}(k) = E\{u(n)u(n-k)\} = \begin{cases} 0, & k \neq 0 \\ \sigma^2, & k = 0 \end{cases}$

$$P_{uu}(\omega) = \sum_{k=-\infty}^{k=\infty} r_{uu}(k)e^{-j\omega k} = \sigma^2$$

$$P_{zz}(\omega) = P_{uu}(\omega)|H(\omega)|^2 = \sigma^2 \frac{1}{1 + a_1^2 + 2a_1 \cos\omega}$$

According to the problem, $\sigma^2 = 0.101043, a_1 = -0.850848$. The plot of the PSD of z(n) is:



Since a_1<0, $P_{zz}(\omega)$ achieves maximum when $\omega = 0$.

b)

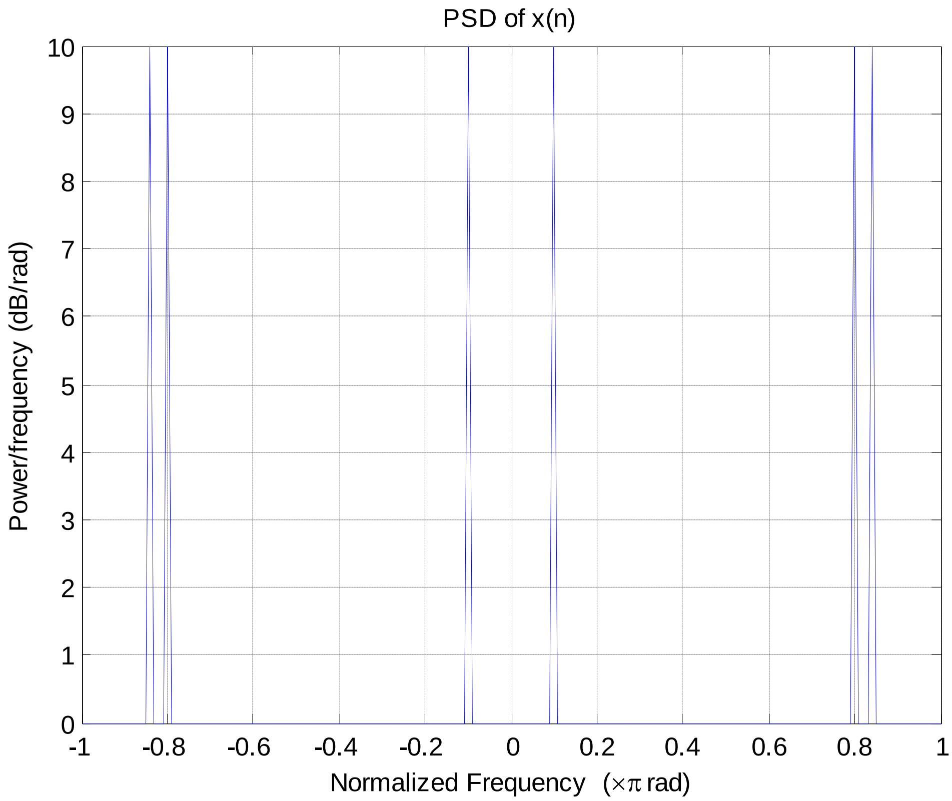
In the absence of z(n), $x(n) = 2\cos(\omega_1 n + \phi_1) + 2\cos(\omega_2 n + \phi_2) + 2\cos(\omega_3 n + \phi_3)$

$\omega = 2\pi f$

$r_{xx}(k) = E\{x(n)x(n-k)\} = 2\cos(\omega_1 k) + 2\cos(\omega_2 k) + 2\cos(\omega_3 k)$

$$P_{xx}(\omega)=\sum_{k=-\infty}^{k=\infty} r_{xx}(k)e^{-j\omega k}=2\pi\{\sum_{k=-\infty}^{k=\infty}\delta(\omega-(\omega_1+2\pi k))+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(-\omega_1+2\pi k))$$
$$+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(\omega_2+2\pi k))+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(-\omega_2+2\pi k))$$
$$+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(\omega_3+2\pi k))+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(-\omega_3+2\pi k))\}$$

The PSD of x(n) consists of a impulse sequence at $\omega=\pm\omega_1+2k\pi$, $\omega=\pm\omega_2+2k\pi$ and $\omega=\pm\omega_3+2k\pi$, $k\in(-\infty,\infty), k\in\mathbb{N}$. It is plotted in $[-\pi,\pi]$ as:



The PSD of x(n) is actually Fourier transform on continues $r_{xx}(t)$ with impulse sampling period T=1.

c)

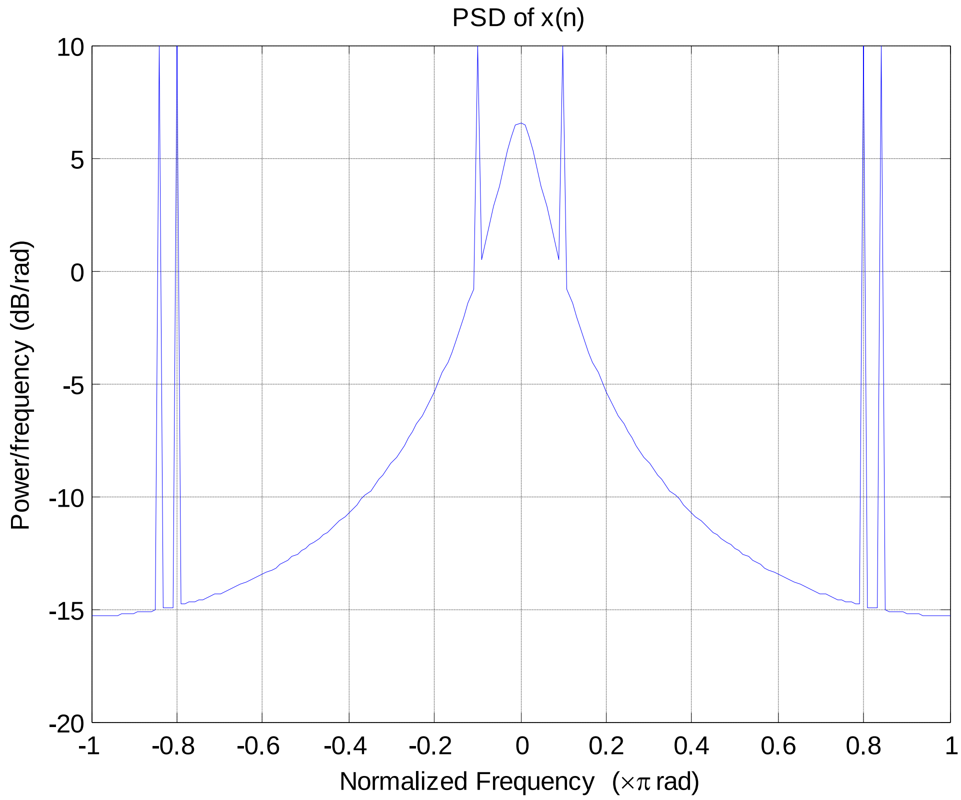
In the presence of z(n), $x(n)=2\cos(\omega_1 n+\phi_1)+2\cos(\omega_2 n+\phi_2)+2\cos(\omega_3 n+\phi_3)+z(n)$

$E\{z(n)\}=-a_1E\{z(n-1)\}+E\{u(n)\}\Rightarrow E\{z(n)\}=0$. So

$r_{xx}(k)=E\{x(n)x(n-k)\}=2\cos(\omega_1 k)+2\cos(\omega_2 k)+2\cos(\omega_3 k)+E\{z(n)z(n-k)\}$

$$P_{xx}(\omega)=\sum_{k=-\infty}^{k=\infty} r_{xx}(k)e^{-j\omega k}=2\pi\{\sum_{k=-\infty}^{k=\infty}\delta(\omega-(\omega_1+2\pi k))+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(-\omega_1+2\pi k))$$
$$+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(\omega_2+2\pi k))+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(-\omega_2+2\pi k))$$
$$+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(\omega_3+2\pi k))+\sum_{k=-\infty}^{k=\infty}\delta(\omega-(-\omega_3+2\pi k))\}+\sigma^2\frac{1}{1+a_1^2+2a_1\cos\omega}$$

The PSD of x(n) is the addition of PSD of x(n) in part b) and the PSD of z(n) in part a). It is plotted as:



2)

a)

5 realizations of ϕ_i are as following:

phi1 =

2.5491 2.5778 2.2171 0.8727 3.7937

phi2 =

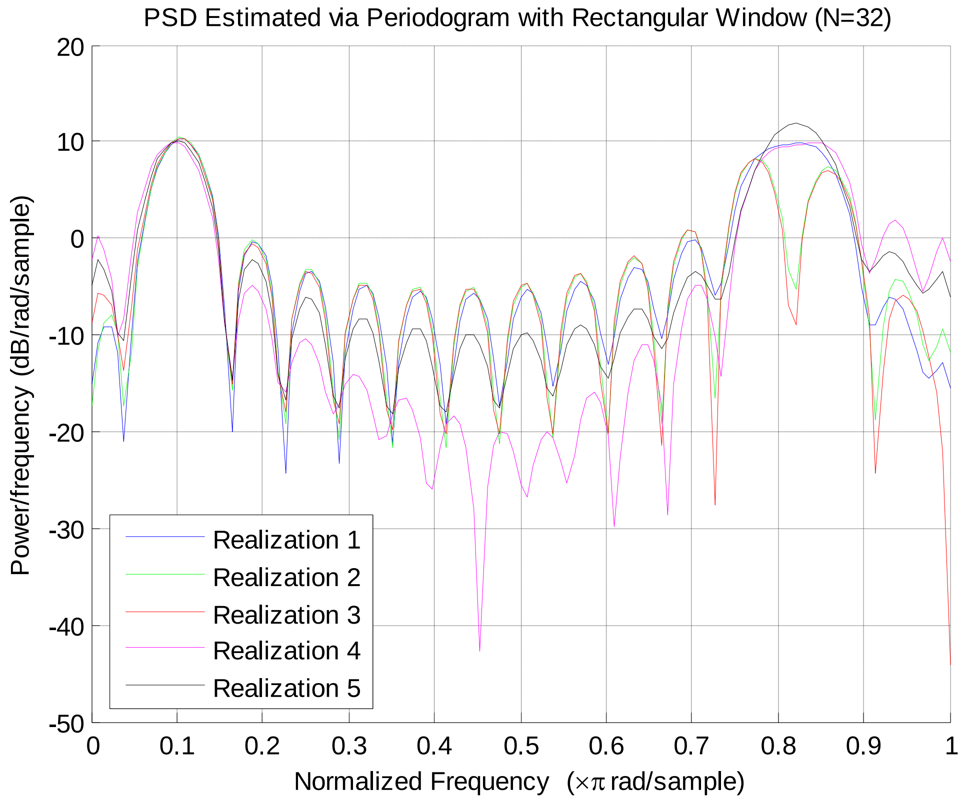
5.8777 5.6150 5.1093 1.2740 1.7102

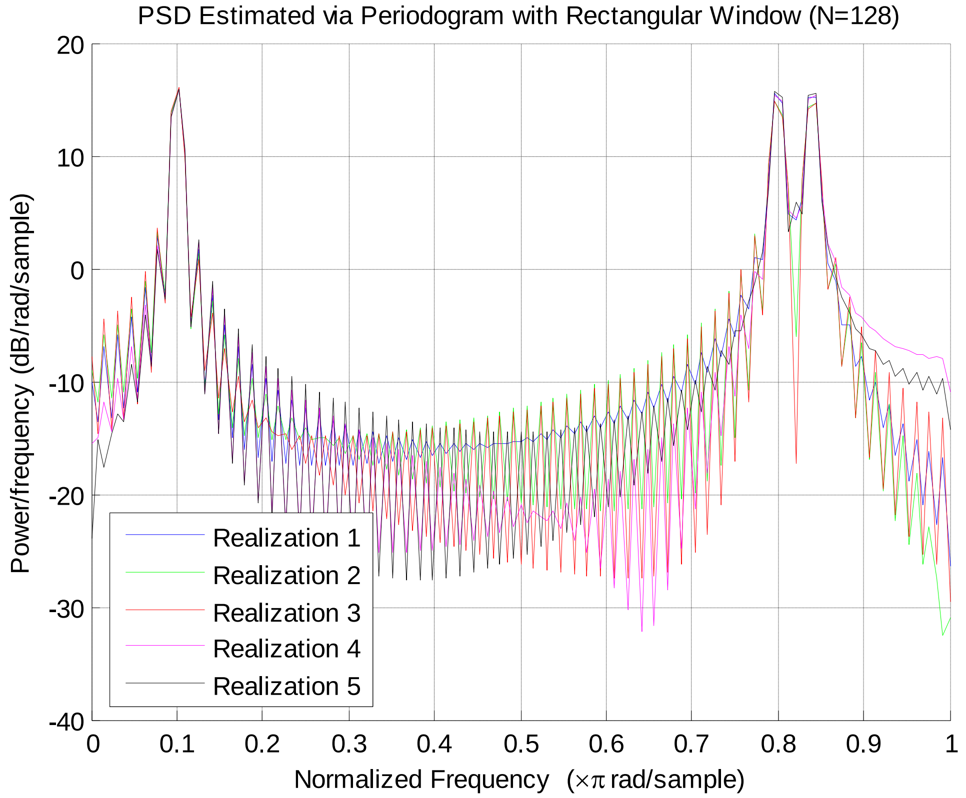
phi3 =

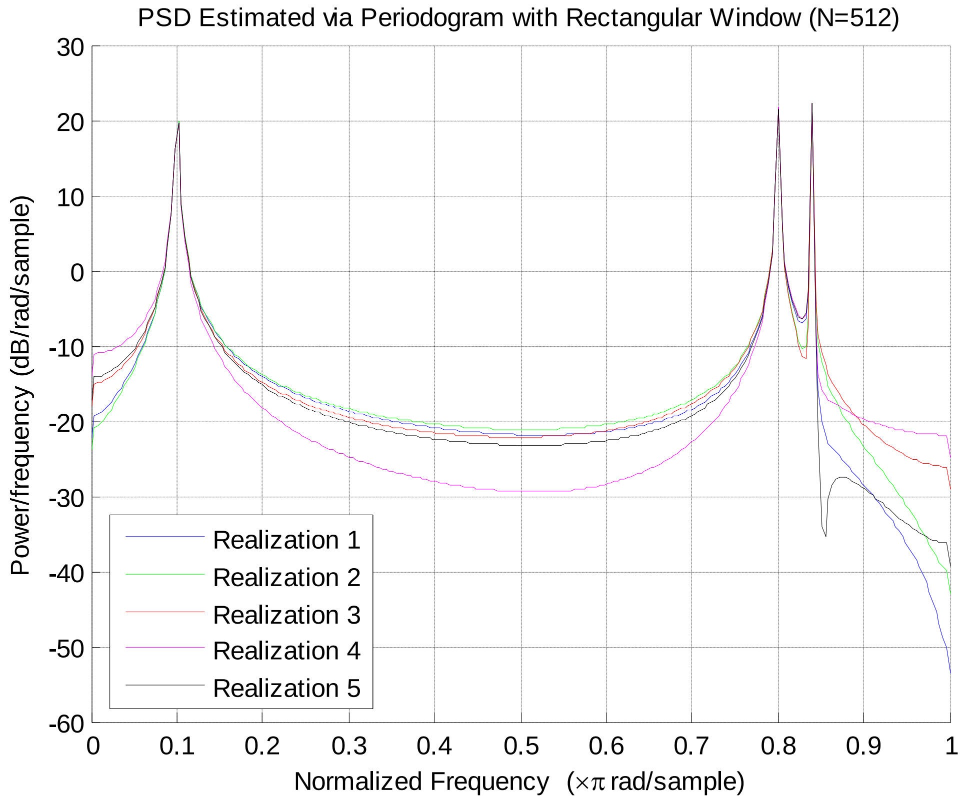
5.7611 0.3637 0.0620 1.2486 1.2492

Since x(n) is real valued sequence, its PSD is symmetric to y-axis, i.e. Pxx(ω)=Pxx(-ω). So we only plot the PSD in the region of $\omega \in [0, \pi]$.

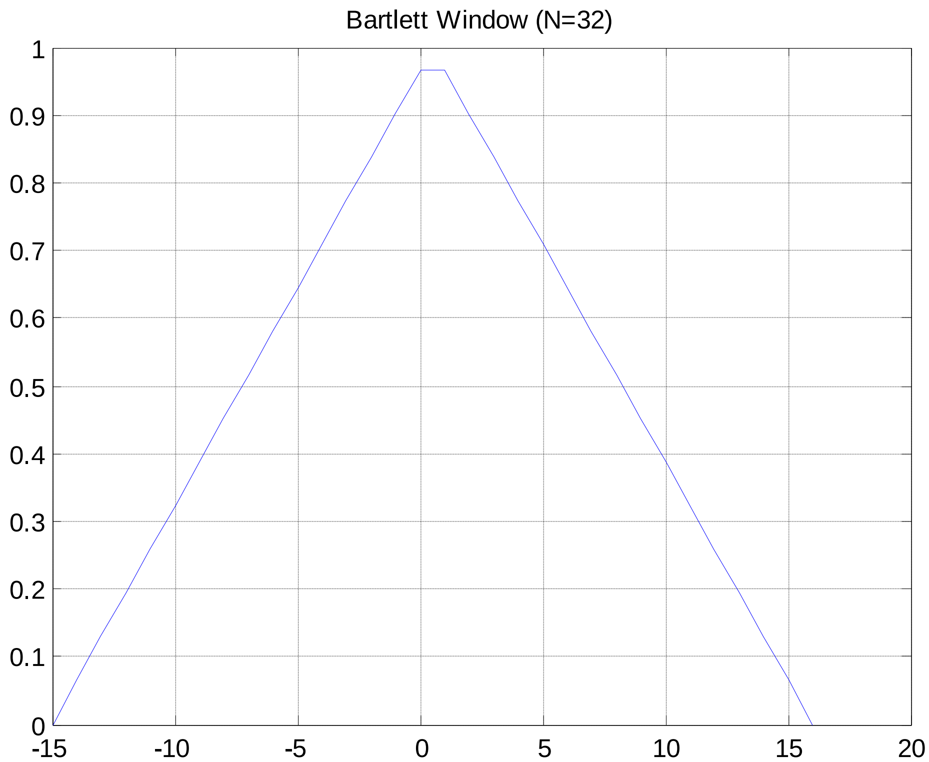
First of all, Periodogram spectral estimation with rectangular window is performed. To estimate the spectral, function periodogram() was called with default value for parameter *nfft*. The plots are given as following:

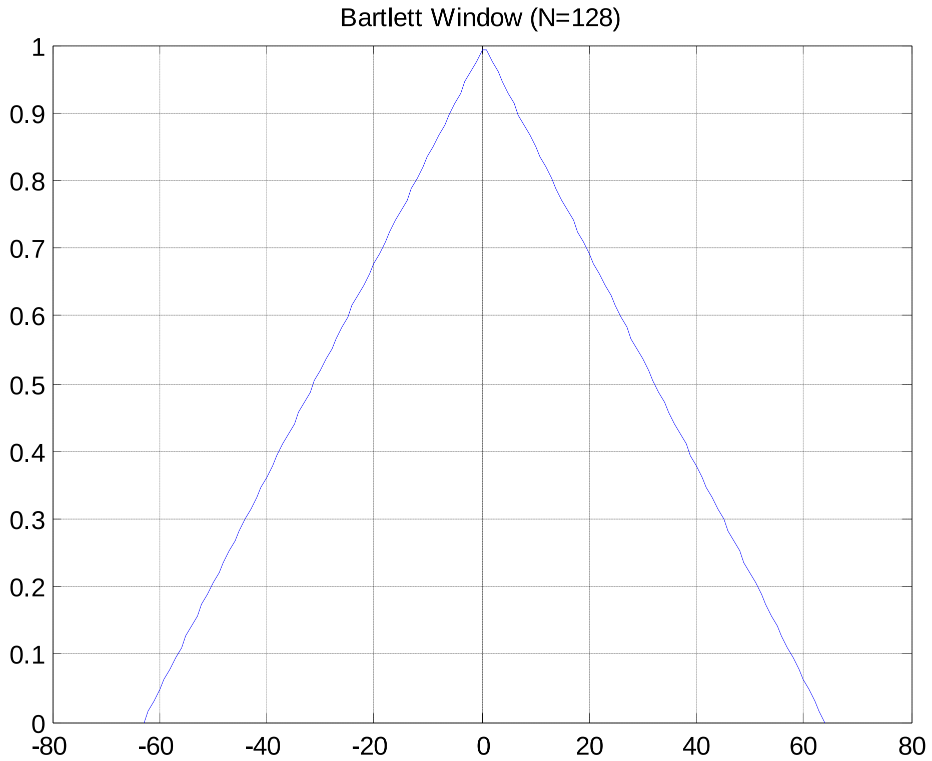

PSD Estimated via Periodogram with Rectangular Window (N=32)
Power/frequency (dB/rad/sample)
20
10
0
-10
-20
-30
-40
-50
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
Realization 1
Realization 2
Realization 3
Realization 4
Realization 5

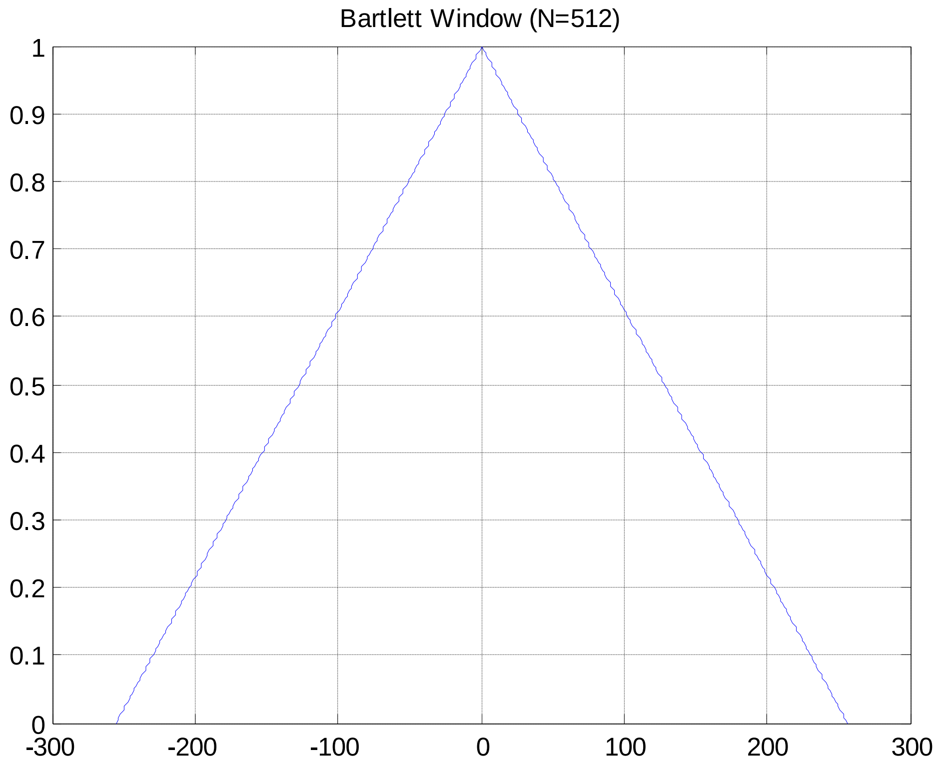

PSD Estimated via Periodogram with Rectangular Window (N=128)
Power/frequency (dB/rad/sample)
20
10
0
-10
-20
-30
-40
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (×π rad/sample)
Realization 1
Realization 2
Realization 3
Realization 4
Realization 5



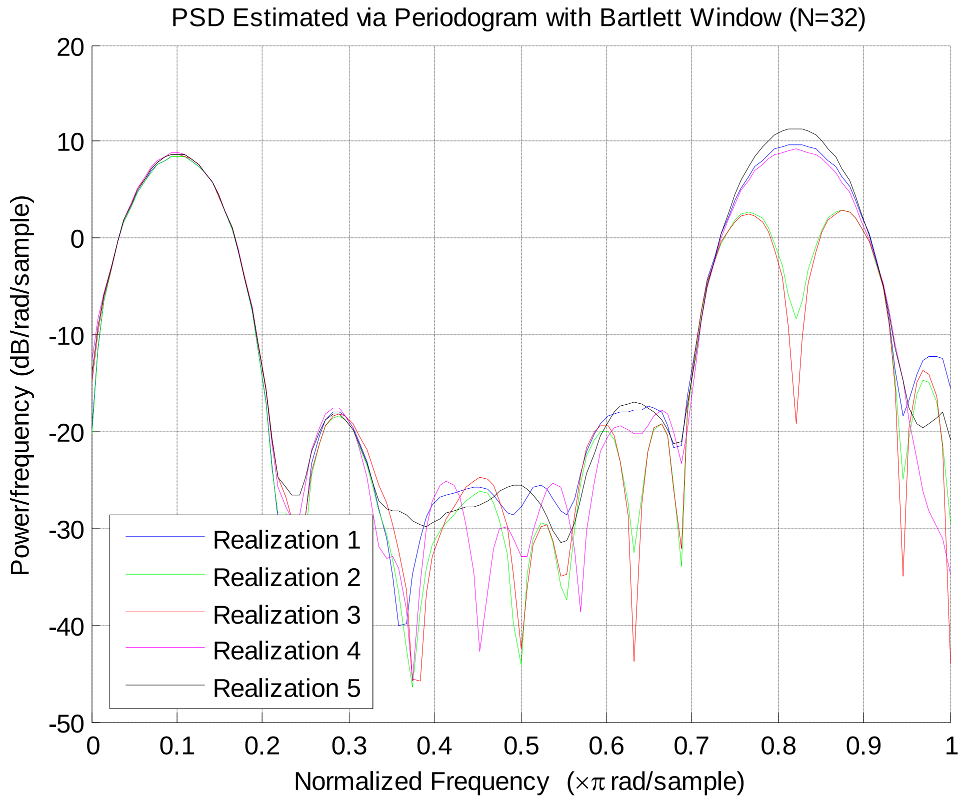
For different values of N, the Bartlett windows are plotted as:

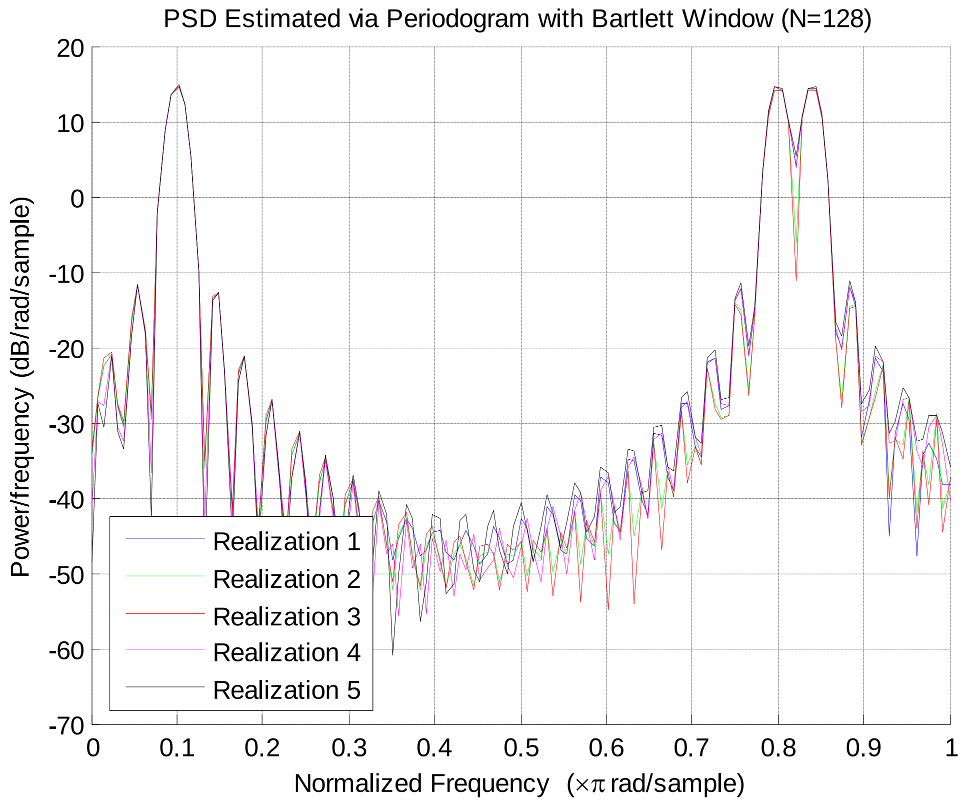


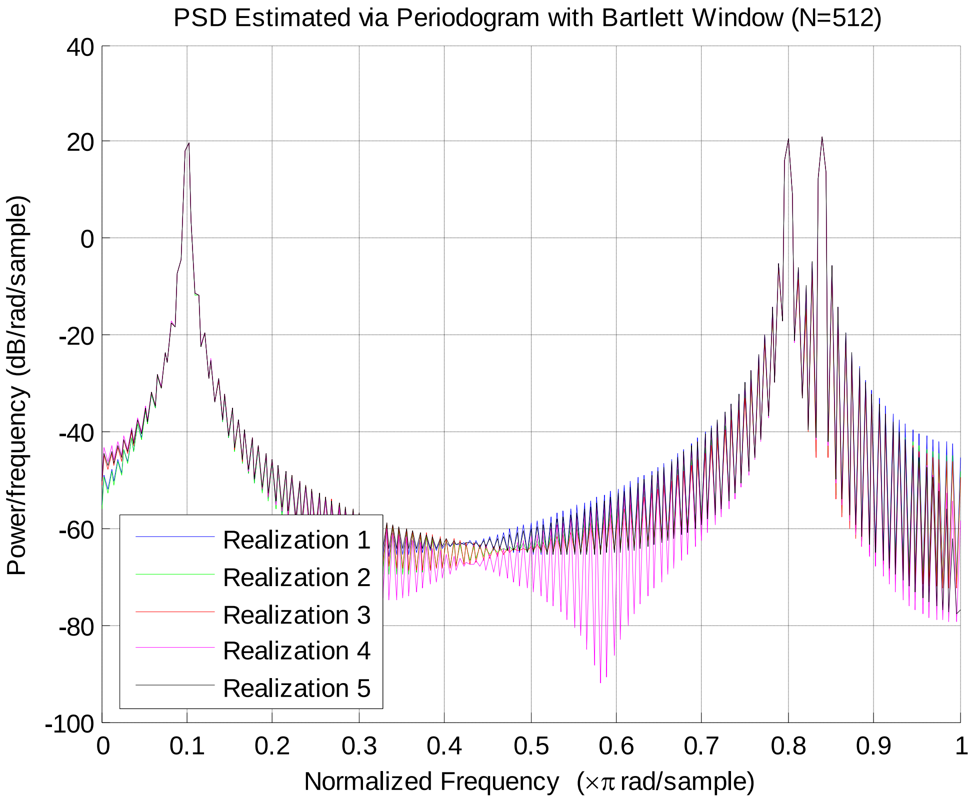




The corresponding PSD estimations are as following:


PSD Estimated via Periodogram with Bartlett Window (N=32)
Power/frequency (dB/rad/sample)
Normalized Frequency (×π rad/sample)
20
10
0
-10
-20
-30
-40
-50
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Realization 1
Realization 2
Realization 3
Realization 4
Realization 5


PSD Estimated via Periodogram with Bartlett Window (N=128)
Power/frequency (dB/rad/sample)
Normalized Frequency (×π rad/sample)
20
10
0
-10
-20
-30
-40
-50
-60
-70
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Realization 1
Realization 2
Realization 3
Realization 4
Realization 5



Discussion:

1. When N increases, the Periodogram estimation of the PSD is getting more accurate and the resolution is getting higher. When N=32, the two peaks at frequencies f2 and f3 are hard to be discriminated. When N=512, they are discriminated very well. It means the bias of Periodogram spectral estimates is decreasing when N is increasing.
2. When N increases, the variance of the Periodogram spectral estimates is still very large.

The resolution limit of periodogram for PSD in frequency domain is 1/N Hz. To tell the two closely spaced sinusoids f2=0.4 Hz and f3=0.42 Hz, we need 1/N <= (f3-f2) which means N >= 50.

This calculation result matches the experimental results well, since for N=32 the two peak at f2 and f3 are not discriminated well but for N=128 and N=512 they are discriminated well.

b)

Yes, we can determine the powers of the sinusoids from the spectral estimates. According to Parseval's energy theorem:

$$\sum_{n=-\infty}^{n=\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) d\omega$$

$S(\omega)$ is the PSD of x(n). Each sinusoid contributes to a peak in the PSD at different frequencies f1, f2 and f3. Suppose the PSD is estimated accurately so that all these peaks are clearly separated. Then we can get the power of the sinusoid (e.g. f1) by integrating $S(\omega)$ within the neighborhood of the corresponding frequency (e.g. ω1=2πf1) and divided by π (not 2π because the sinusoid contributes to two peaks at region [0, π] and [-π, 0] respectively).

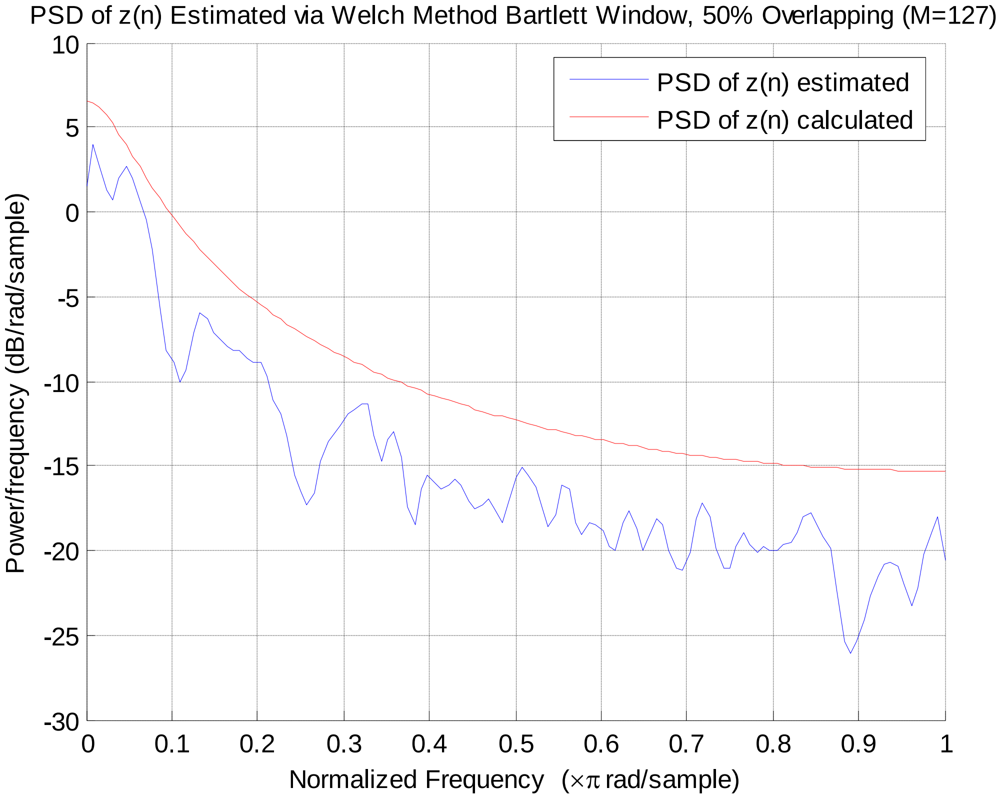
3).

a)

First of all, we need to generate the initial value of z(n). When $N \to \infty$, we assume that z(n) is stationary. And the initial value of z(n) is $z(0) \sim N(0, r(0))$, and

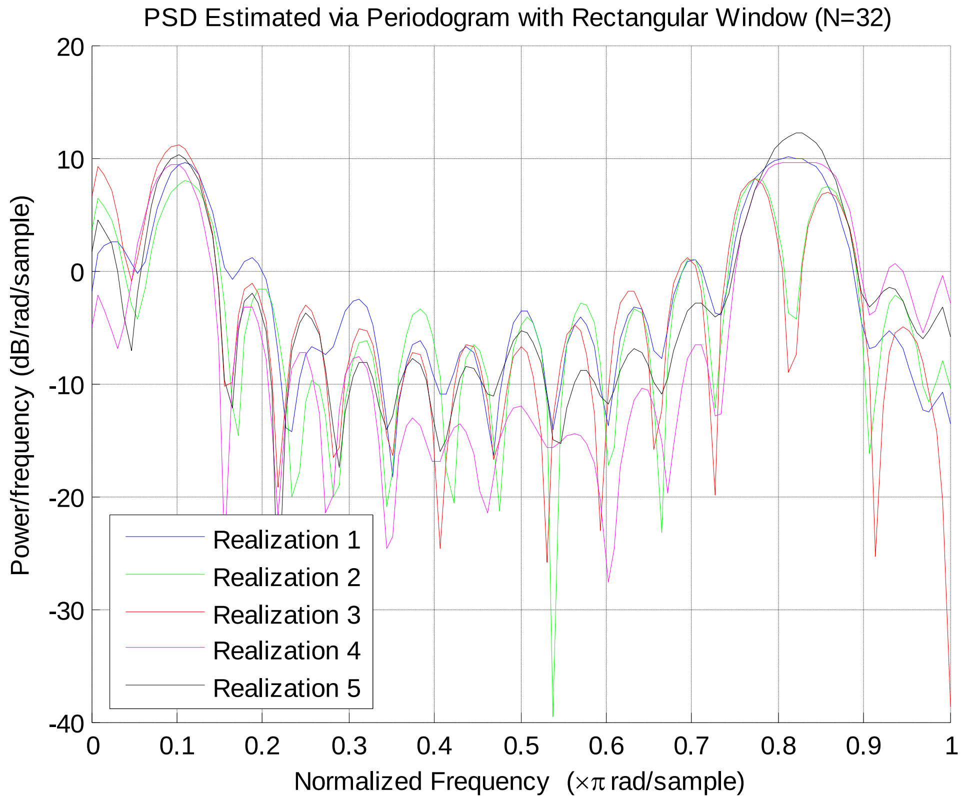
$$r(0) = E\{z(n)^2\} = a_1^2 E\{z(n-1)^2\} - 2a_1 E\{z(n-1)u(n)\} + E\{u(n)^2\} = a_1^2 r(0) + \sigma^2$$

So we have $r(0) = \dfrac{\sigma^2}{1-a_1^2} = 0.3660$. So we can use z(0)=sqrt(0.3660)*randn(1). Following plots shows the matching of the PSD estimated from a realization of z(n) to the calculated PSD of z(n):

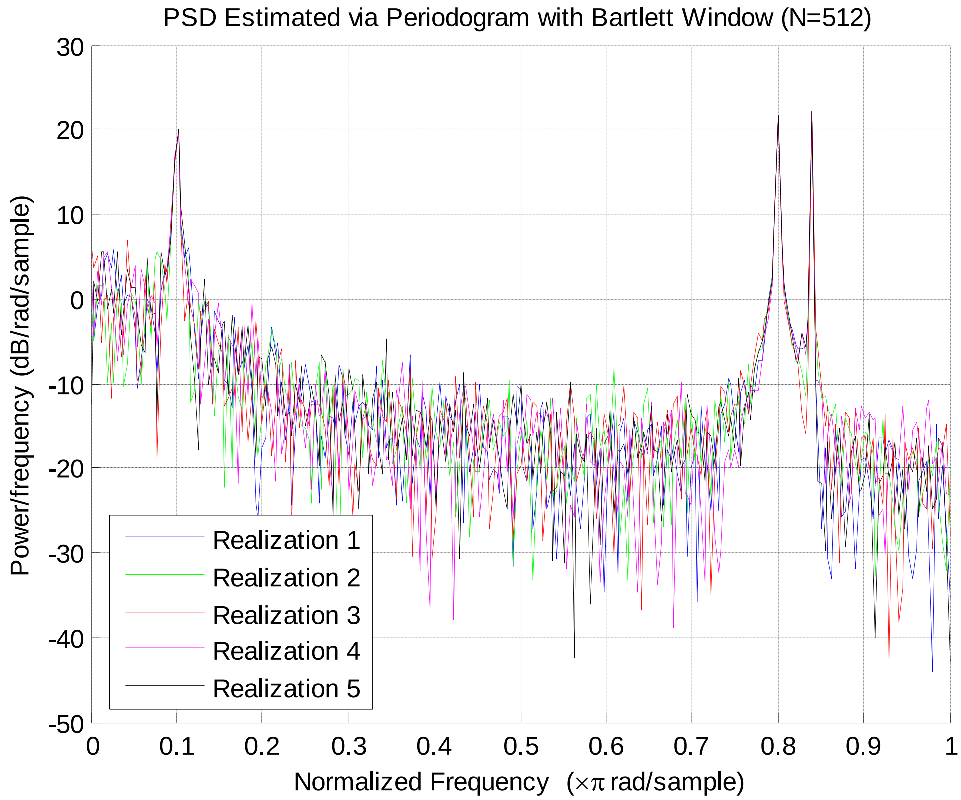


The choices of ϕ are the same as that in problem 2 which do not change over different methods. But the noises are realized differently in different methods. Following are the plots of the estimated PSD using different methods.

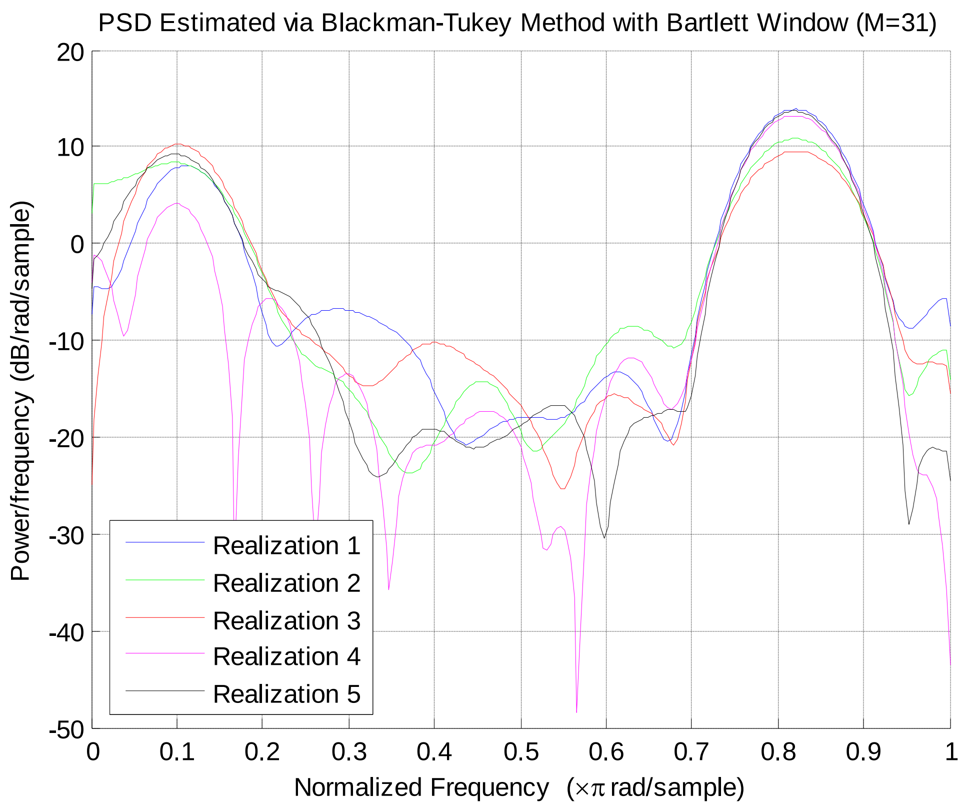
i) N=32, Periogram



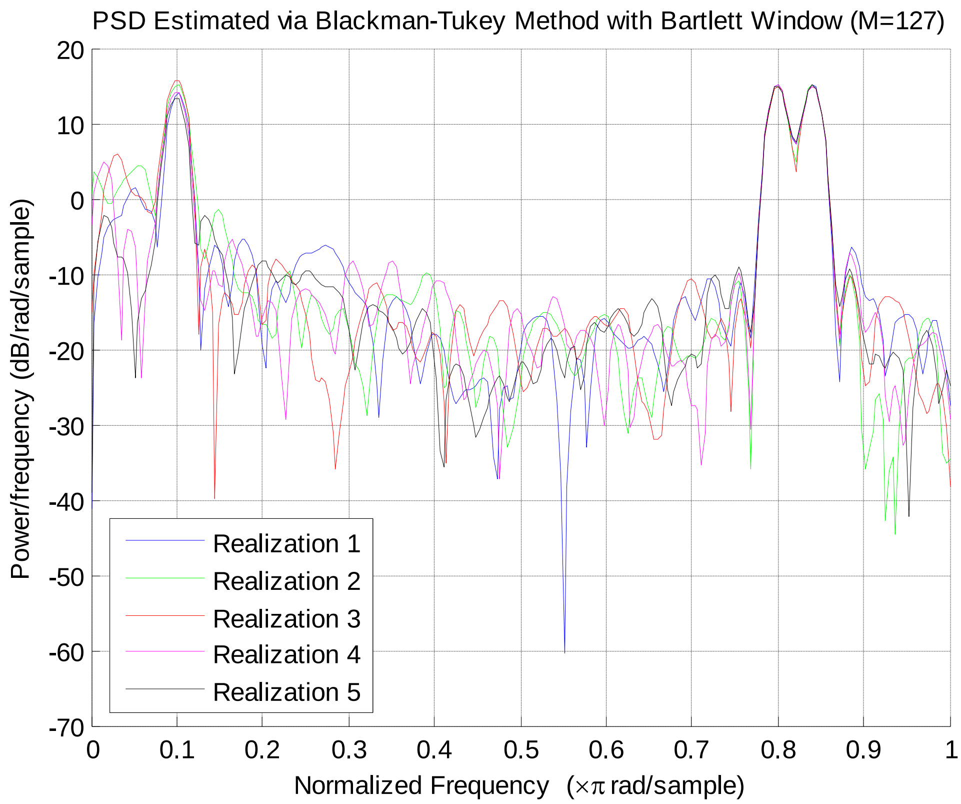
ii) N=512, Periogram



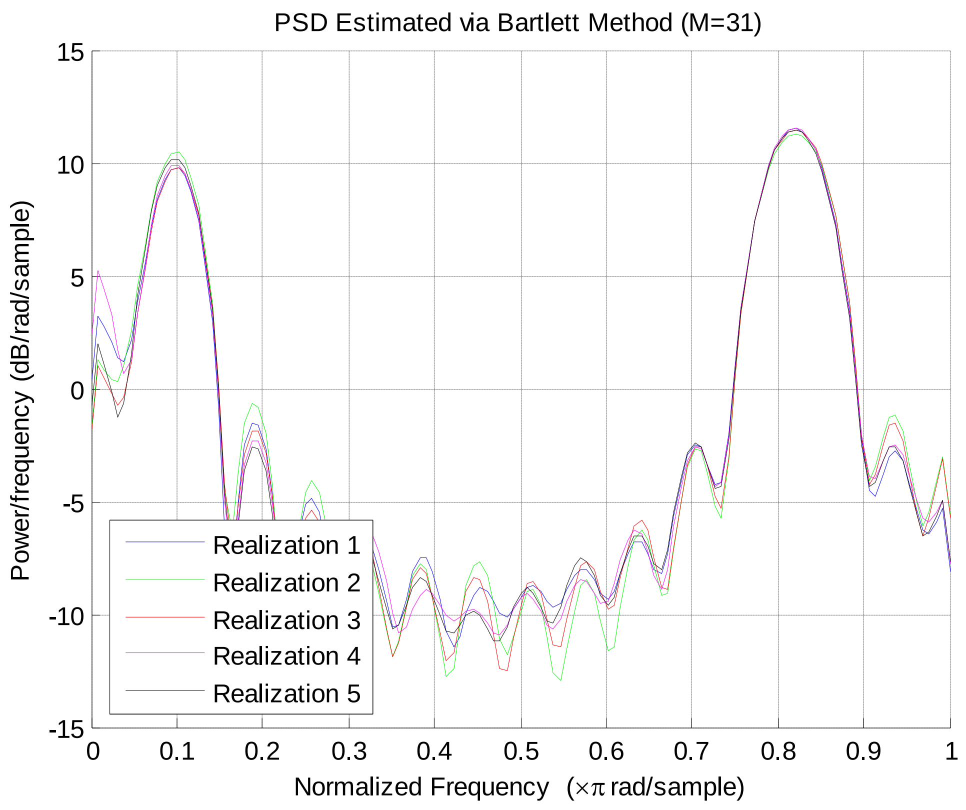
iii) N=512, Blackman-Tukey Method and Bartlett window with M=31.

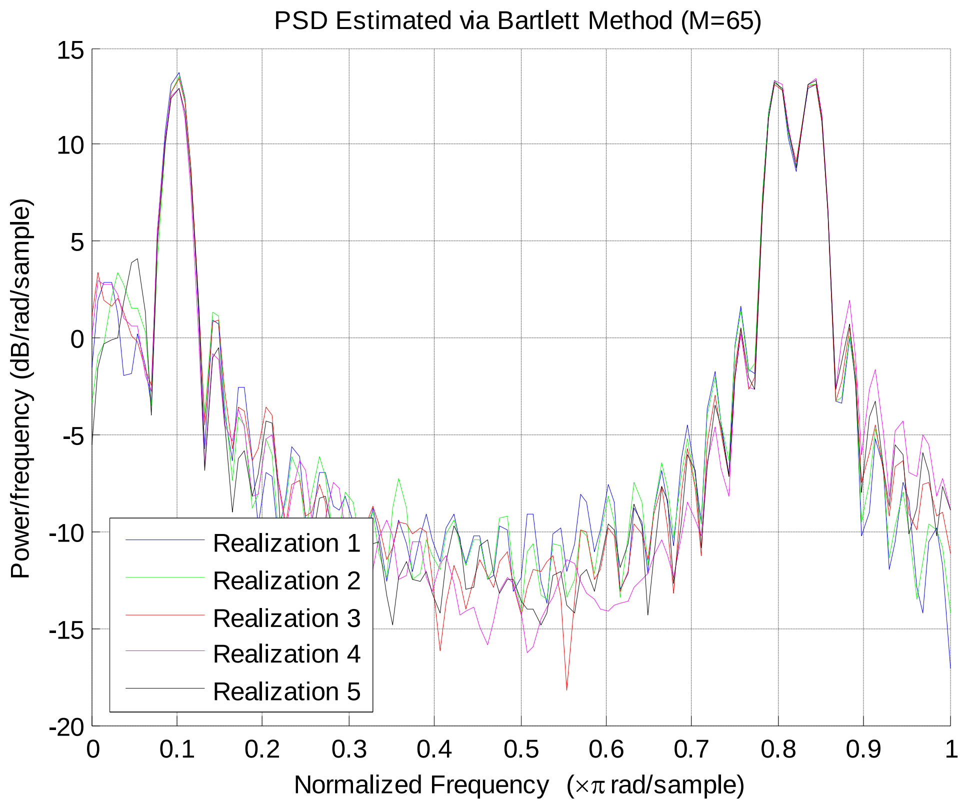


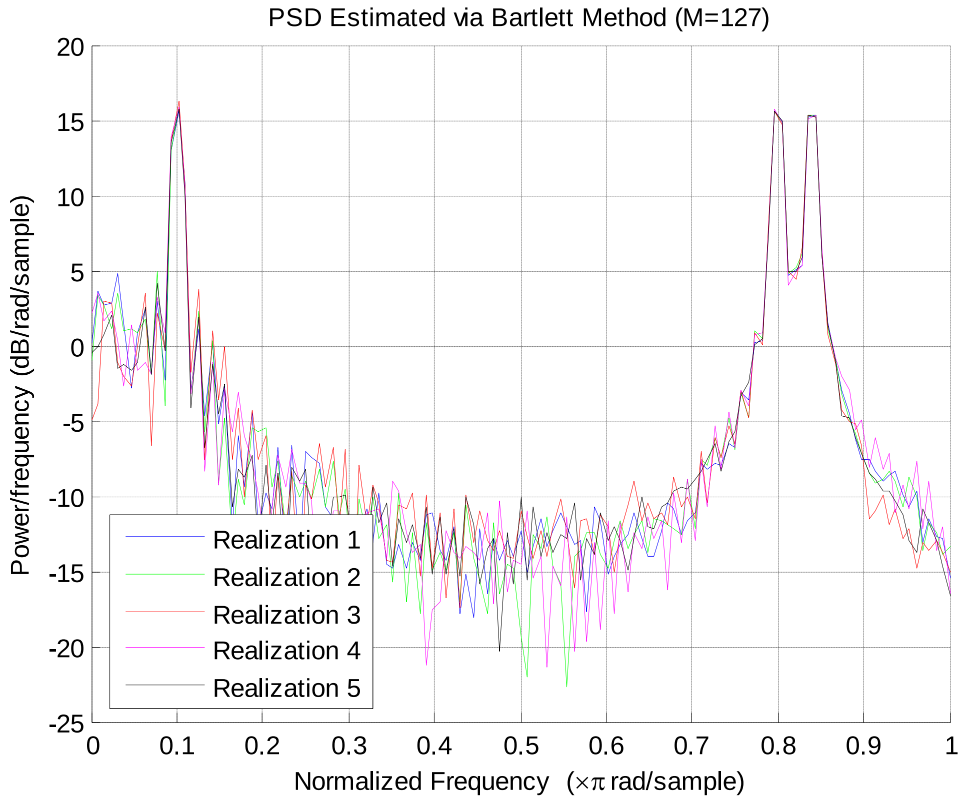
iv) N=512, Blackman-Tukey Method and Bartlett window with M=127.



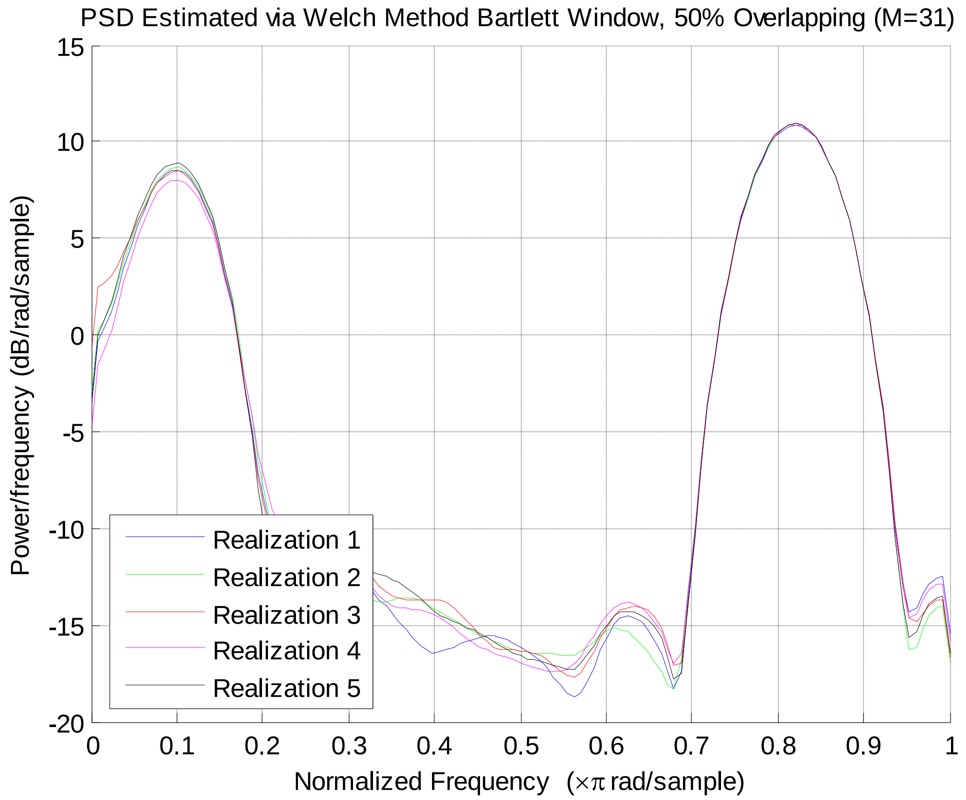
v) N=512, Bartlett Method with M=31, 65, 127

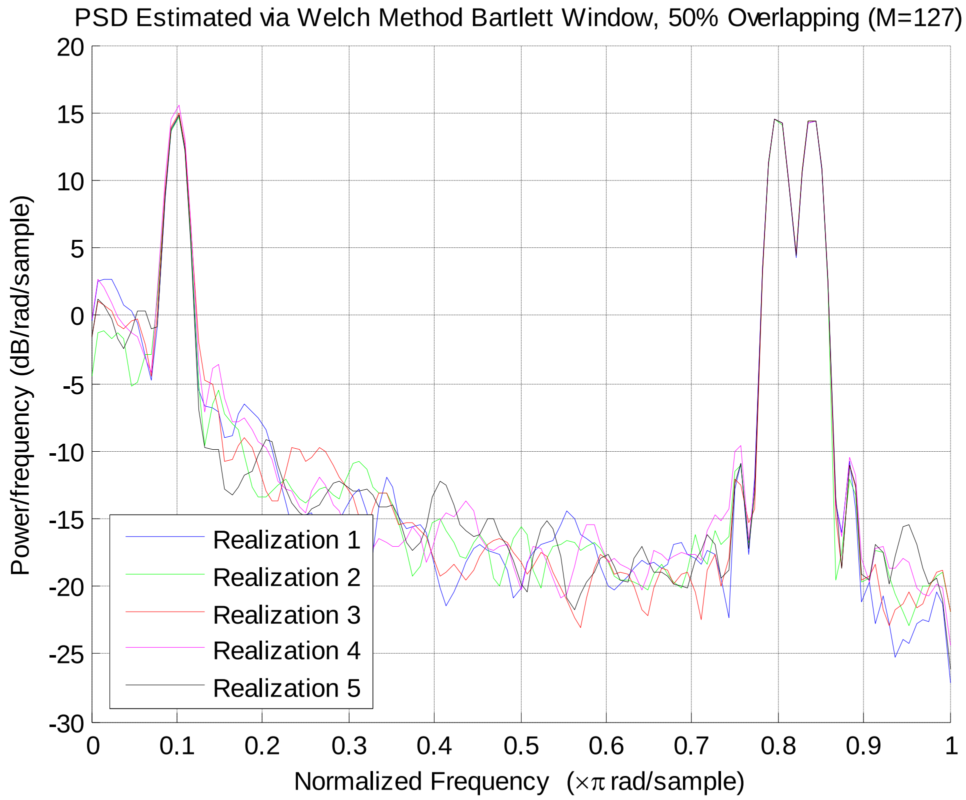

PSD Estimated via Bartlett Method (M=31)
Power/frequency (dB/rad/sample)
Normalized Frequency (×π rad/sample)
Realization 1
Realization 2
Realization 3
Realization 4
Realization 5
15
10
5
0
-5
-10
-15
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


PSD Estimated via Bartlett Method (M=65)
Power/frequency (dB/rad/sample)
Normalized Frequency (×π rad/sample)
Realization 1
Realization 2
Realization 3
Realization 4
Realization 5
15
10
5
0
-5
-10
-15
-20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

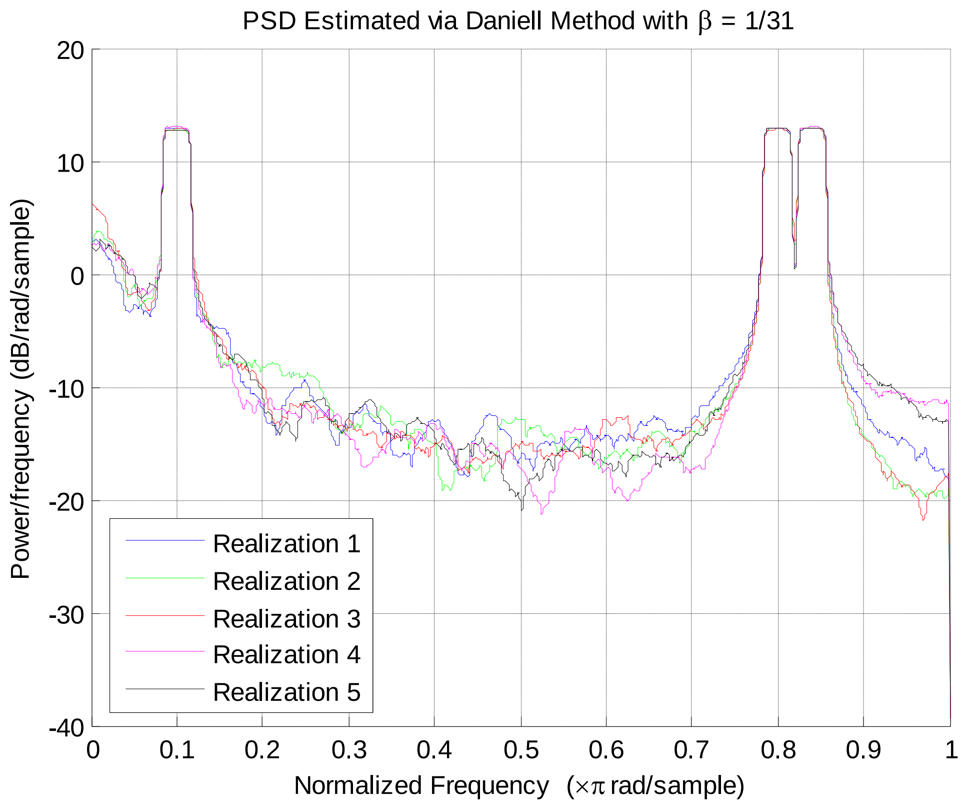


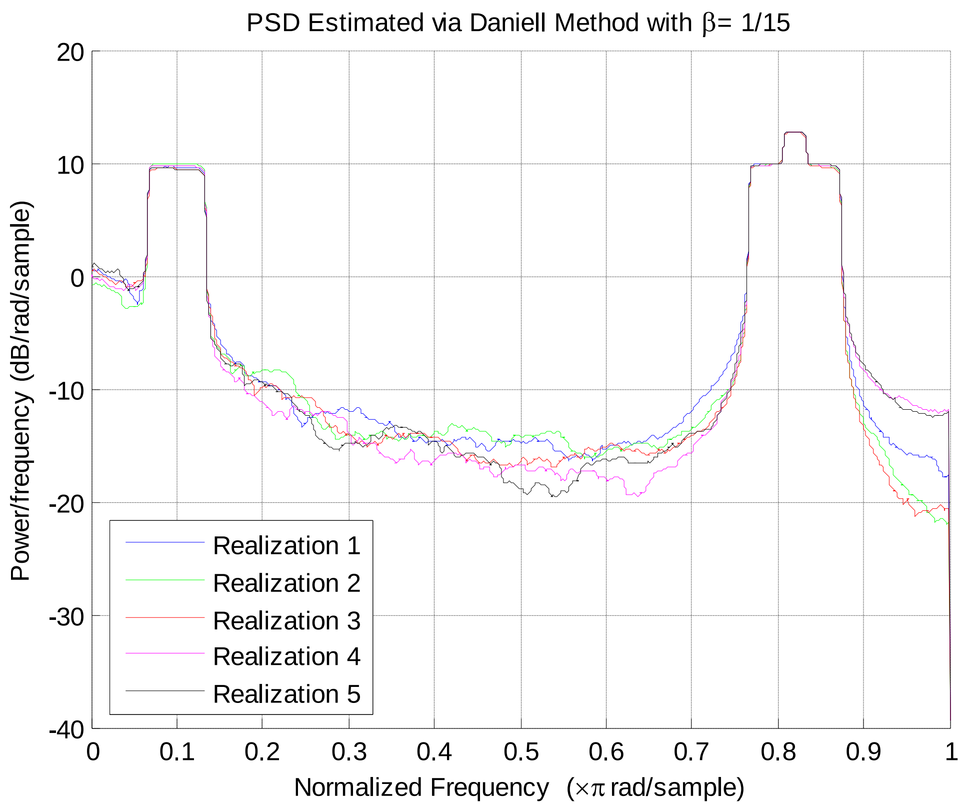
vi) N=512, Welch Method with M=31, 50% overlapping and Bartlett window

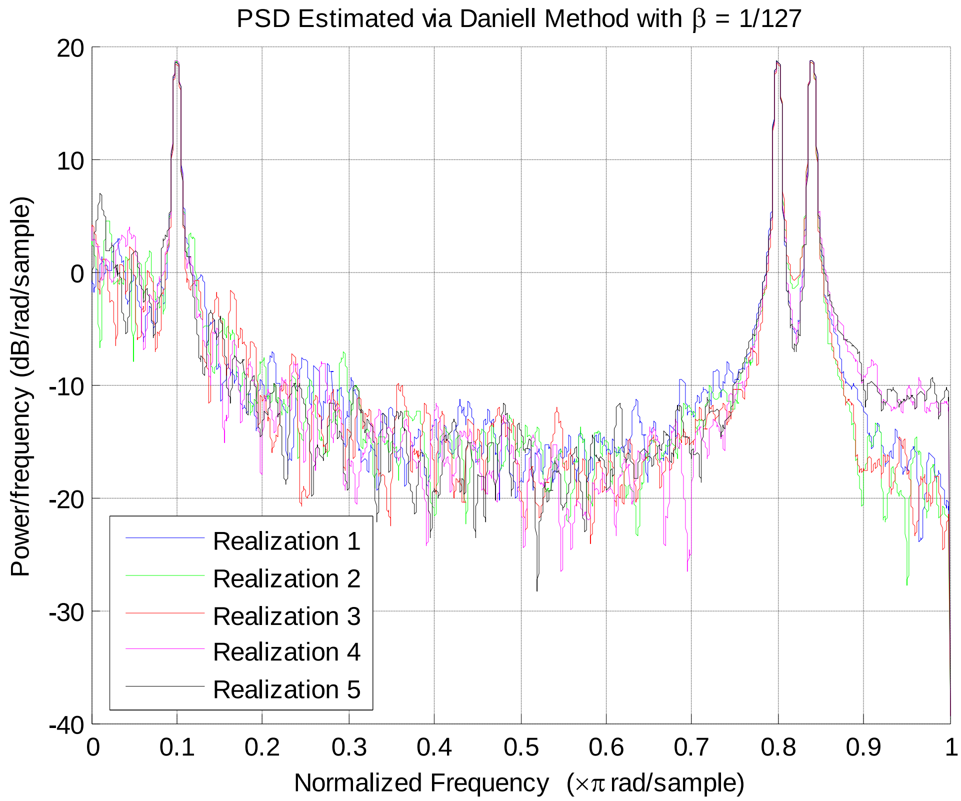




vii) N=512, Daniel Method with β=1/31, 1/15, 1/127.







b) Discussion

1. For the basic Periogram method, when N increases, resolution increases while bias decreases. The variance of the estimates does not decrease. According to the analysis, when

N is large enough, the bias of Periodogram estimation can be eliminated. But the problem of Periodogram is large variance.

2. Compared with Periodogram method, Blackman-Tukey method has larger bias, lower resolution and smaller variance. At the same time, when M decreases, the variance becomes larger and the resolution becomes lower. The reason is that the main lobe of the smoothing window in the frequency domain becomes wider.

3. Compared with basic Periodogram method, Bartlett method has lower resolution and smaller variance. When N is fixed and M is increasing, the resolution is getting higher and the variance is also getting larger.

4. Welch method also has lower resolution and smaller variance. Its variance seems smaller than the variance of Bartlett method. But its resolution seems lower than that of Bartlett method.

5. Daniel method also has lower resolution and smaller variance compared with basic Periodogram method. The tradeoff between resolution and variance is controlled by β.

6. The resolution and variance is a pair of tradeoff. All the modified methods are trying to decrease the large variance by scarifying resolution or bias.