Optimally tapered periodograms for nonuniformly sampled data

Zhaofu Chen\textsuperscript{1} Prabhu Babu\textsuperscript{2} Petre Stoica\textsuperscript{3} Jian Li\textsuperscript{4}
\textsuperscript{1,4}Department of Electrical and Computer Engineering, University of Florida
Gainesville, FL, USA, 32611
\textsuperscript{2,3}Division of Systems and Control, Department of Information Technology
Uppsala University, P.O. Box 337, Uppsala, SWEDEN, SE-751 05
\textsuperscript{1}cgcruiser@ufl.edu
\textsuperscript{2}prabhu.babu@it.uu.se
\textsuperscript{3}ps@it.uu.se
\textsuperscript{4}li@dsp.ufl.edu
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Abstract
In this paper, we design tapers for the periodogram in the case of nonuniformly sampled data. We consider both data independent and data dependent taper designs. In the data independent design case, we derive tapers by minimizing either the integrated sidelobe level or the peak sidelobe level of the spectral window. In the data dependent case, we extend the design by allowing the tapers to be complex-valued, as well as frequency and data dependent and obtain them by iterative techniques. Finally, we evaluate the performance of the designed tapers via several numerical examples.

1 Introduction
Spectral estimation plays an important role in the area of signal processing and communications. Accurate spectral estimators from a finite number of uniformly placed samples are usually the first block of various signal processing algorithms. A classical method for spectral estimation is the periodogram, which computes the spectrum by taking the squared magnitudes of the Fourier transform of the data. Although the raw periodogram suffers from poor resolution and leakage problems, its wide spread use is attributed to its low complexity and easy implementation via the computationally efficient fast Fourier transforms (FFT). The resolution and leakage problems of the periodogram are usually addressed by
tapering or windowing the data (we will use the term “taper” and “window” interchangeably) before calculating the periodogram, and the performance of the taper can be directly linked to the mainlobe width and sidelobe levels in the corresponding spectral window. The wider the mainlobe of the spectral window, the poorer the resolution of the periodogram; similarly the higher the sidelobe levels of the spectral window, the larger the leakage in the periodogram. We refer to [1] for more details on the window design for the periodogram in the uniform sampling case.

However in some cases the data samples are observed at nonuniform time intervals. For example in astronomy [2] [3], seismology [4], paleoclimatology [5], genetics [6], nuclear quadrupole resonance spectroscopy [7] and laser Doppler velocimetry [8], the observations are acquired at nonuniform time instants owing to the unavailability of the data during certain instants of time or due to some constraints on the hardware resources. The periodogram for nonuniform data can be obtained in a similar fashion as in the uniform data case by directly taking the squared magnitude of the Fourier transform calculated at different frequencies. However in addition to the smearing and leakage problems due to finite sample length, the periodogram of non-uniform data may exhibit some spurious peaks which are mainly due to data nonuniformity.

In this paper, we obtain optimally tapered periodograms for nonuniform data by solving certain optimization problems. The tapers obtained here are classified into data independent tapers and data dependent tapers. In the former category, we obtain two data independent tapers (DIT), DIT1 and DIT2, by minimizing the integrated sidelobe level and the peak sidelobe level of the spectral window, respectively. In the latter category, we obtain two data dependent tapers (DDT), DDT1 and DDT2, by means of certain iterative algorithms. DDT1 based method happens to coincide with the recently developed iterative adaptive approach (IAA) algorithm [9] [10], whereas DDT2 can be interpreted as a regularized Version of IAA which we will call VIAA.

This paper is organized as follows. Section 2 provides the preliminaries and the notations used in the paper. In section 3, we derive all the data independent and data dependent tapers. Section 4 provides the numerical examples illustrating the performances of DITs and DDTs, and section 5 ends the paper with the conclusions.
2 Notations and preliminaries

Let the continuous time signal \( y(t) \) be measured at a set of possibly nonuniform time indices \( \{t_n\}_{n=1}^N \).

Then the periodogram of the data sequence is given by:

\[
\hat{\Phi}(\omega) = |\hat{\alpha}(\omega)|^2 \\
\hat{\alpha}(\omega) = \frac{1}{N} \sum_{n=1}^{N} y(t_n)e^{-j\omega t_n}
\]  

(1)

where \( \omega \) is the frequency variable. Note that \( \hat{\alpha}(\omega) \) above can be interpreted as an estimate of the amplitude spectrum of the data, and also that (1) can be implemented efficiently using the nonuniform fast Fourier transform (NFFT) \([11]\) \([12]\). Usually the periodogram is evaluated on a uniform grid

\[
\omega = p\Delta \omega, \quad p = -P, \ldots, P - 1.
\]  

(2)

The choice of \( \Delta \omega \) and \( P \) in (2) mainly depends on \( \{t_n\}_{n=1}^N \) via the so-called spectral window:

\[
\left| \frac{1}{N} \sum_{n=1}^{N} e^{-j\omega t_n} \right|^2.
\]  

(3)

The 3dB width of the mainlobe of (3) is approximately equal to \( \frac{2\pi}{(t_N-t_1)} \) \([1]\) \([13]\). Consequently \( \Delta \omega \) should be chosen smaller than \( \frac{2\pi}{(t_N-t_1)} \). For example, \( \Delta \omega = \frac{2\pi}{10(t_N-t_1)} \). Regarding the choice of \( P \), we note that (3) takes on its maximum value of 1 at \( \omega = 0 \); \( P \) is chosen such that \( 2P\Delta \omega \) is less than or equal to the smallest frequency (\( \neq 0 \)) at which (3) takes on a value close to 1 (see \([14]\) \([15]\) \([16]\) for details). (If there is no such a frequency, then \( P = \infty \), at least in theory.)

3 Optimal taper designs

Consider the following weighted least squares (WLS) problem:

\[
\min_{\alpha(\omega)} \sum_{n=1}^{N} w_n |y(t_n) - \alpha(\omega)e^{j\omega t_n}|^2
\]  

(4)

with \( w_n \geq 0 \) for all \( n \) and \( \sum_{n=1}^{N} w_n = 1 \). The minimizer of (4) is given by:

\[
\hat{\alpha}_T(\omega) = \sum_{n=1}^{N} w_n y(t_n)e^{-j\omega t_n}
\]  

(5)
where the subscript \( T \) indicates an estimate of \( \alpha(\omega) \) obtained from tapered data. The condition that the weights should be nonnegative is necessary since (4) is a weighted least squares problem, and the condition that the weights should add up to one is imposed to simplify (5) by avoiding the division with \( \sum_{n=1}^{N} w_n \). The resulting spectral estimate, which is commonly called a tapered periodogram, is given by:

\[
\hat{\Phi}_T(\omega) = |\hat{\alpha}_T(\omega)|^2 = \left| \sum_{n=1}^{N} w_n y(t_n)e^{-j\omega t_n} \right|^2.
\]  

(6)

The estimate \( \hat{\alpha}(\omega) \) in (1) can be obtained from (4) by using \( w_n = \frac{1}{N} \) for all \( n \). The spectral window in the case of tapering is given by:

\[
\left| \sum_{n=1}^{N} w_n e^{-j\omega t_n} \right|^2.
\]  

(7)

We might think of choosing the taper in (6) as:

\[
w_n = \frac{1}{\sigma_n^2},
\]  

(8)

where \( \sigma_n^2 \) is the variance of the error in the observation \( y(t_n) \). With (4) in mind, this taper makes sense from a statistical viewpoint. However, reliable information on \( \sigma_n^2 \) is not frequently available in practice, and therefore using (8) may not be possible in general.

The WLS criterion in (4) with

\[
\left\{ \begin{array}{l}
w_n = \frac{\bar{w}_n}{\sum_{n=1}^{N} \bar{w}_n}, \quad n = 1, \ldots, N \\
\bar{w}_n = \frac{t_{n+1} - t_{n-1}}{2}, \quad n = 2, \ldots, N - 1 \\
\bar{w}_N = t_N - t_{N-1}, \quad \bar{w}_1 = t_2 - t_1
\end{array} \right.
\]  

(9)

can be interpreted as a discrete approximation (to within a multiplicative constant) to the following integral least squares criterion:

\[
\int |y(t) - \alpha(\omega)e^{j\omega t}|^2 dt.
\]

This interpretation provides some motivation to (9) except for the situations in which there are relatively large gaps in the sampling time sequence \( \{t_n\} \). The taper in (9) will be referred to as the integral approximation taper (IAT) in the subsequent sections of the paper.

Many other tapers can be borrowed from the case of uniformly sampled data. For example, the
following Hanning-type taper, named shortly here as HT

\[
\begin{align*}
\begin{cases}
w_n &= \frac{\bar{w}_n}{\sum_{n=1}^{N} w_n}, & n = 1, \ldots, N \\
\bar{w}_n &= 0.5 \left[1 - \cos \left(\frac{2\pi(t_n - t_1)}{t_N - t_1}\right)\right], & n = 1, \ldots, N
\end{cases}
\end{align*}
\]  

(10)

was used in [17], and it was claimed to possess some form of suboptimality in [18] from a sidelobe control standpoint. The intuitive motivation for using (10) is to taper off the end parts of the data sequence to reduce the “discontinuity” induced by the finiteness of the sequence; however, the problems induced by the non-uniformity of the data (such as large gaps) are not taken care of by (10).

It follows from (7) that the leakage from a possible frequency \( p\Delta \omega \), in the data sequence, to the frequency \( k\Delta \omega \) is proportional to:

\[
\left|\sum_{n=1}^{N} w_n e^{j\Delta \omega (p-k)t_n}\right|^2, \quad p, k = -P, \ldots, P - 1.
\]

(11)

Depending on how we formulate the problem of minimizing (11) (for \( p \neq k \)) with respect to \( \{w_n\} \), we get different types of data independent tapers, as explained in what follows.

3.1 Data independent taper designs

3.1.1 Data independent taper 1 (DIT1)

The values taken by \( p - k \) in (11) belong to \([-2P + 1, 2P - 1]\). With \( l = p - k \), the integrated criterion of the squared magnitudes of the mainlobe and sidelobes is given by:

\[
\sum_{l=-2P+1}^{2P-1} \left|\sum_{n=1}^{N} w_n e^{j\Delta \omega lt_n}\right|^2 = 2 \sum_{l=0}^{P-1} \left|\sum_{n=1}^{N} w_n e^{j\Delta \omega lt_n}\right|^2 - 1 = 2 \sum_{l=-P}^{P-1} \left|\sum_{n=1}^{N} w_n e^{j\Delta \omega lt_n}\right|^2 - 1
\]

(12)

the first equality follows from the fact that the criterion is even and the second equality follows from the fact that the spectral window is periodic with period equal to \( 2P\Delta \omega \). On dropping the scaling factor and the additive constant, we get

\[
\sum_{l=-P}^{P-1} \left|\sum_{n=1}^{N} w_n e^{j\Delta \omega lt_n}\right|^2.
\]

(13)

The above criterion includes the mainlobe of the spectral window, which is typically much larger than the sidelobes; hence minimization of (13) will emphasize the reduction of the width of the mainlobe, even
by small amounts, and will pay little attention to the sidelobe magnitudes. To exclude the mainlobe, we modify (13) as follows:

\[
\sum_{l \in [-P, -L] \cup [L, P-1]} \left| \sum_{n=1}^{N} w_n e^{j \Delta \omega l t_n} \right|^2
\]

for a given \(L\). A typical value of \(L\) is 5, whose use excludes the mainlobe from the criterion (the width of the mainlobe is close to \(\frac{2\pi}{|t_N - t_1|} = 10\Delta \omega\)). The optimal taper in DIT1 is obtained by minimizing the integrated sidelobe criterion in (14):

\[
\min \left\{ w_n \right\} \sum_{l \in J} \left| \sum_{n=1}^{N} w_n e^{j \Delta \omega l t_n} \right|^2,
\]

s.t. \(w_n \geq 0, \ n = 1, \ldots, N\)

\[
\sum_{n=1}^{N} w_n = 1
\]

where \(J = [-P, -L] \cup [L, P-1]\). Let \(A\) denote the matrix with elements given by:

\[
A_{n, \bar{n}} = \sum_{l \in J} e^{j \Delta \omega l (t_n - t_{\bar{n}})} , \ n, \bar{n} = 1, \ldots, N
\]

Then (15) can be rewritten as:

\[
\min_{w} \ w^T \text{Re}(A) w
\]

s.t. \(w_n \geq 0, \ n = 1, \ldots, N\)

\[
w^T u = 1.
\]

where \(w = [w_1, \ldots, w_N]^T\), \(u = [1, \ldots, 1]^T\), and \(\text{Re}(A)\) denotes the real part of \(A\). This is a convex quadratic program (QP) that can be efficiently solved [19] [20]. We note that this way of obtaining tapers in the nonuniform data case is similar to the prolate-spheroidal window (also known as Slepian sequence) design in the uniform data case [1], where one designs the taper by maximizing the ratio of the power confined in the mainlobe of the spectral window to the total power. Once the optimal taper \(w\) is obtained, the spectral estimate at any frequency \(\omega\) can be calculated from (6).

### 3.1.2 Data independent taper 2 (DIT2)

Instead of the integrated sidelobe criterion used above, we can use the following minimax criterion:

\[
\min \left\{ w_n \right\} \max_{l \in J} \left| \sum_{n=1}^{N} w_n e^{j \Delta \omega l t_n} \right|^2,
\]

s.t. \(w_n \geq 0, \ n = 1, \ldots, N\)

\[
\sum_{n=1}^{N} w_n = 1.
\]
The optimization problem above can be rewritten as (here \( \beta \) is an auxiliary variable):

\[
\min_{\beta, \{w_n\}} \beta
\]

s.t.
\[
\left| \sum_{n=1}^{N} w_n e^{j \Delta \omega t_n} \right|^2 \leq \beta, \ \forall l \in J
\]
\[w_n \geq 0, \ n = 1, \ldots, N\]
\[
\sum_{n=1}^{N} w_n = 1
\]

(19)

which is a second-order cone program (SOCP) that can be efficiently solved \([19][20]\). Once the optimal taper is obtained from (19), the spectrum can be obtained by evaluating (6). The optimization problem in (19) is similar to the Dolph-Chebsyhev taper design problem in the uniform data case \([21]\). By design, the maximum sidelobe of the spectral window corresponding to DIT2 will be smaller than that of DIT1. However, many other sidelobes (there may be up to \( N \) sidelobes equal to the maximum one) may be larger than the corresponding sidelobes of DIT1, which can be a drawback in some applications.

Let us consider a data set containing 40 non-uniform samples with sampling indices as shown in Fig.1. Also shown is the spectral window of the sampling pattern, from which we can see that the spectrum repeats itself every \( 2\pi \) rads/sec. Thus the parameters \( \Delta \omega \) and \( P \) which are used in the design can be chosen as:

\[
\Delta \omega = \frac{2\pi}{10(t_N - t_1)}
\]
\[
P = \frac{\pi}{\Delta \omega}.
\]

(20)

Figure 1: (a) Sampling pattern and (b) Spectral window.

The data is made of two complex sinusoids corrupted by zero-mean white Gaussian noise with variance equal to 3. The frequencies of the two sinusoids are taken to be 0.2Hz and 0.4Hz, with amplitudes 10 and 5 respectively. The initial phases of the sinusoids are equal to zero. Fig. 2 shows the spectra of
the data obtained from the untapered periodogram, the periodograms of the data tapered by DIT1 and
DIT2, and the periodograms of the data tapered by the heuristic tapers IAT and HT. The circles in
the plots indicate the true power and frequencies of the two sinusoids present in the data. It is seen
from Fig. 2 that although the spectrum obtained using the tapers DIT1 and DIT2 indicate the presence
of the weak frequency component at 0.4Hz, their performance, in terms of the sidelobe level, is not so
significant when compared to the heuristic tapers and unwindowed periodogram. Moreover, unlike the
uniform sampling case, the width of the mainlobe is not directly proportional to $L$. In fact, sometimes
increasing the value of $L$ leads to splitting of mainlobe into many small lobes which is undesirable in
some applications.

In the next subsection, we relax the taper design problem by assuming that the taper can be complex-
valued, as well as frequency and data dependent. In doing so we leave the framework based on the spectral
window but gain more degrees of freedom that will allow achieving a more significant improvement in
the spectrum estimation problem.

### 3.2 Data dependent taper designs

#### 3.2.1 Data dependent taper 1 (DDT1)

The estimate $\hat{\alpha}(\omega)$ in (1), for $\omega = k\Delta\omega$, can be rewritten as:

$$
\hat{\alpha}(k) = \sum_{n=1}^{N} h_n(k)y(t_n)
$$

for $h_n(k) = w_ne^{-j\omega kt_n}$. 

(21)

The constraint $\sum_{n=1}^{N} w_n = 1$ can be written in terms of $\{h_n(k)\}$ as:

$$
\sum_{n=1}^{N} h_n(k)e^{j\omega kt_n} = 1
$$

(22)

In what follows, we allow $\{h_n(k)\}$ to be complex-valued numbers, so we do not constrain them to depend
on $\{w_n\}$ as above (with $w_n \geq 0$). Then the WLS interpretation of $\hat{\alpha}(\omega)$ in (4) is lost. However, the
so-obtained estimate $\hat{\alpha}(\omega)$ still makes a lot of sense, as we explain below. Note that the amplitude
spectral estimate frequency in (21) can be written more compactly as:

$$
\hat{\alpha}(k) = h^*(k)y,
$$

$$
h^*(k) = [h_1(k) \ldots h_N(k)],
$$

$$
y = [y(t_1) \ldots y(t_N)]^T
$$

(23)
Figure 2: Power spectrum of the data considered in section 3.1, obtained from: (a) Unwindowed periodogram (b) DIT1 tapered periodogram ($L = 5$) (c) DIT2 tapered periodogram ($L = 5$) (d) IAT tapered periodogram (e) HT tapered periodogram (SNR = 16dB).
and the constraint in (22) as:

\[
\begin{align*}
\mathbf{h}^*(k)\mathbf{a}(k) &= 1, \\
\mathbf{a}(k) &= [e^{j\Delta\omega_1k}, \ldots, e^{j\Delta\omega_Nk}]^T
\end{align*}
\]  

(24)

where \( \mathbf{h}^*(k) \) denotes the conjugate transpose of the vector \( \mathbf{h}(k) \). Hence the \( \hat{\alpha}(k) \) above is a linear estimate (linear in the data vector \( y \)) and \( \mathbf{h}(k) \) can still be considered, by a slight abuse of terminology, to be a taper. The generalized taper \( \mathbf{h}(k) \), which is complex valued and frequency dependent, can be obtained such that it passes the frequency of interest undistorted (see (24)) and suppresses all other possible frequency components (see (25) below). The taper design problem can thus be written as:

\[
\begin{align*}
\min_{\mathbf{h}(k)} \quad &\mathbf{h}^*(k)\hat{\mathbf{A}}\mathbf{h}(k) \\
\text{s.t.} \quad &\mathbf{h}^*(k)\mathbf{a}(k) = 1
\end{align*}
\]  

(25)

where \( \hat{\mathbf{A}} = \sum_{p=-P}^{P-1} \mathbf{a}(p)\mathbf{a}^*(p) \). Assuming that \( \hat{\mathbf{A}} \succ 0 \), the solution to (25) can be computed in closed form (e.g. [1]):

\[
\mathbf{h}(k) = \frac{\hat{\mathbf{A}}^{-1}\mathbf{a}(k)}{\mathbf{a}^*(k)\hat{\mathbf{A}}^{-1}\mathbf{a}(k)}
\]  

(26)

For “pathological sampling patterns”, \( \hat{\mathbf{A}} \) will be nearly rank deficient, which will result in (26) having too large a “noise gain” \( ||\mathbf{h}(k)||^2 \). In such cases, \( \hat{\mathbf{A}} \) in (25) can be replaced by \( \hat{\mathbf{A}} + \rho\mathbf{I} \), where the diagonal loading factor \( \rho \) can be varied until a design with a reasonable noise gain is obtained, a typical value of \( \rho \) is 0.001. The taper in (26) is somewhat similar to DIT1 designed with \( L = 0 \), as the matrix \( \hat{\mathbf{A}} \) in (26) is identical to the matrix \( \mathbf{A} \) (with \( L = 0 \)) in (16). We know from the data independent taper design that even DIT1 with careful choice of \( L \) cannot decrease the sidelobe levels significantly; in addition to that the design criterion in (25) includes the mainlobe of the spectral window, so the performance of the taper in (26) may be poorer than that of DIT1.

If a priori information \( \hat{\Phi}(k\Delta\omega) \) on the spectral power is available, we can replace \( ||\mathbf{h}^*(k)\mathbf{a}(p)||^2 \) in the design criterion by \( \hat{\Phi}(p)\mathbf{h}^*(k)\mathbf{a}(p)\mathbf{a}^*(p) \) to obtain a data dependent taper; in other words the minimization problem in (25) can be replaced by

\[
\begin{align*}
\min_{\mathbf{h}(k)} \quad &\mathbf{h}^*(k)\mathbf{R}\mathbf{h}(k) \\
\text{s.t.} \quad &\mathbf{h}^*(k)\mathbf{a}(k) = 1
\end{align*}
\]  

(27)

where \( \mathbf{R} = \sum_{p=-P}^{P-1} \hat{\Phi}(p)\mathbf{a}(p)\mathbf{a}^*(p) \), and where we can take \( \hat{\Phi}(p) = ||\mathbf{h}^*(p)y||^2 \). Since the matrix \( \mathbf{R} \) depends on \( \mathbf{h}(k) \), the minimization problem in (27) has to be solved iteratively. Specifically, at the \( i^{th} \) iteration,
the taper at frequency $k$ is obtained analytically as

$$h_i(k) = \frac{R_{i-1}^{-1}a(k)}{a^*(k)R_{i-1}^{-1}a(k)} (28)$$

where $R_{i-1} = \sum_{p=-P}^{P} \hat{\Phi}^{-1}_{i-1}(p)a(p)a^*(p)$, with $\hat{\Phi}^{-1}_{i-1}(p) = |h^*_{i-1}(k)y|^2$. The corresponding amplitude spectral estimate at frequency $k$ is given by

$$\hat{\alpha}_i(k)y = \frac{a^*(k)R_{i-1}^{-1}y}{a^*(k)R_{i-1}^{-1}a(k)} (29)$$

which is exactly the estimate obtained by the recently developed iterative adaptive approach (IAA) [9] [10].

The IAA spectrum of the data set considered in subsection 3.1 is shown in Fig. 3; it is seen from the figure that the IAA spectrum has very low sidelobe levels and the frequency component at 0.4Hz is clearly visible. However for pathological sampling patterns, the $R$ matrix used in IAA becomes rank deficient and hence non-invertible. For instance, when the sampling pattern shown in Fig. 4 is used for the data model of two complex sinusoids discussed in subsection 3.1, IAA fails to recover the spectrum, see Fig.5(a).

![Figure 3: IAA spectrum of the data considered in section 3.1 (compare with Fig. 2).](image)

\subsection{3.2.2 Data dependent taper 2 (DDT2)}

In this section, we develop a regularized taper named DDT2 to tackle pathological sampling schemes. We will call DDT2 as VIAA as it can be viewed as a (generalized) version of IAA. VIAA differs from IAA in the construction of $R$ but otherwise VIAA is based on the same iterative process as IAA, see [9]
Figure 4: Example of a pathological sampling pattern (a) Sampling pattern and (b) Spectral window.

Figure 5: Power spectrum of the data in section 3.1 sampled using the pattern in Fig. 4 (a) IAA and (b) VIAA (SNR = 16dB).
The $R$ matrix used in VIAA is constructed as follows:

$$R = \sum_{p \in \mathcal{M}} \hat{\Phi}(p) a(p)a^*(p) + \sum_{p \in [-P,P-1] \setminus \mathcal{M}} \hat{\Phi}(p) I$$

where $\mathcal{M}$ denotes the subset of $[-P,P-1]$ containing the indices corresponding to the $N$ largest values of $\hat{\Phi}(p)$ and $I$ denotes the identity matrix of size $N \times N$. The reasons for constructing $R$ as in (30) are as follows:

1. For pathological sampling patterns, the $R$ in IAA will be nearly singular. However $R$ in (30) will virtually always be non-singular; the first term in (30) ensures that we construct $R$, an $N \times N$ matrix, from the Fourier vectors corresponding to the $N$ largest values in the spectrum, and the second term (constructed from the rest of the values in the spectrum) acts as a regularization term.

2. The second reason is based on the fact that from $N$ data samples we can reliably estimate no more than $N$ values in the spectrum. So we use the $N$ largest values in the spectrum to construct the “signal part of” $R$ and the sum of the remaining values in the spectrum is taken as an estimate of the noise floor.

To test VIAA for a pathological sampling pattern, we used the sampling scheme in Fig. 4: VIAA recovered the spectrum quite accurately as shown in Fig.5(b). We have also compared VIAA with an existing regularization technique for IAA named IAAR [22], and found that VIAA has a faster convergence rate than IAAR’s.

4 Numerical Examples

In this section, we evaluate the performance of the tapers derived in Section 3 on two data sets. In Example 1, we consider a high resolution application which consists of resolving two closely spaced sinusoids in noise. In Example 2, we consider a real life data set containing radial velocity measurements for the star HD 208487, whose spectrum is expected to have a peak at a frequency corresponding to the period of rotation of a possible planet revolving around the star.

4.1 Example 1

The data considered in this example is made of 40 nonuniform samples of two complex sinusoids with frequencies 0.2 Hz and 0.21 Hz and amplitudes equal to 10. The sampling pattern used here is same as that used in section 3.1 (see Fig. 1). The noise in the data is complex white Gaussian noise with zero mean and variance equal to 3. Fig. 6 shows the plots of the periodograms obtained by using different
tapers, including the unwindowed periodogram. For DIT1, the value of $L$ is chosen to be 0 to achieve the highest possible resolution. In the case of DIT2, choosing $L = 0$ gives an undesirable result of all sidelobe levels equal to the mainlobe level, so we chose a slightly higher value of $L = 5$. As can be seen from this figure all tapers, except HT, resolve the peaks; this is especially true for IAA and VIAA which have very low sidelobe levels compared to the other methods.

4.2 Example 2

The data set considered in this example consists of 35 nonuniform samples of radial velocity measurements for the star HD 208487. The sampling pattern and its corresponding spectral window are shown in Fig. 7. Fig. 8 shows the periodograms obtained by different tapering techniques, including the unwindowed periodogram. The values of $L$ for DIT1 and DIT2 are chosen somewhat arbitrarily to be 10 and 15 respectively. It can be seen from Fig. 8 that except VIAA all the other methods indicate the presence of several peaks, which are mainly due to sampling artifacts. Also IAA fails in this example due to the pathological sampling pattern. For VIAA it is seen clearly that the spectrum has a peak at 0.066 cycles/day, which suggests that the period of the exo-planet revolving around the star is 15.15 days.

5 Conclusion

In this paper, we designed periodogram tapers in the case of nonuniform sampling. The tapers designed here were grouped under two broad categories: data independent and data dependent tapers. In the data independent taper design case, the tapers were obtained by minimizing two criteria derived from the spectral window, namely integrated sidelobe level and peak sidelobe level. For data dependent designs, we relaxed the problem and assumed that the tapers can be complex-valued, as well as frequency and data dependent. In the latter case, the tapers were obtained by iterative techniques and they have superior performance compared to the data independent tapers. Finally, we evaluated the performance of the designed tapers by carrying out several numerical simulations on artificial and experimental data sets.

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Figure 6: Power spectrum of the data considered in Example 1 obtained from (a) Unwindowed periodogram (b) DIT1 tapered periodogram ($L = 0$) (c) DIT2 tapered periodogram ($L = 5$) (d) IAT tapered periodogram (e) HT tapered periodogram (f) IAA and (g) VIAA (SNR = 18.0dB).
Figure 7: (a) Sampling pattern and (b) Spectral window of the radial velocity data.

Figure 8: Power spectrum of the radial velocity data obtained from (a) Unwindowed periodogram (b) DIT1 tapered periodogram \((L = 10)\) (c) DIT2 tapered periodogram \((L = 15)\) (d) IAT tapered periodogram (e) HT tapered periodogram (f) VIAA.
References


