Homework 4

EEL6533

Due date: October 11, 2011

1. A special case of the binary Gaussian problem with $N$ observations is

$$f(y|H_i) = \frac{1}{(2\pi)^{N/2} |\Sigma_i|^{1/2}} \exp \left[ -\frac{1}{2} y^T \Sigma_i^{-1} y \right], \quad i = 0, 1.$$  

a) Find $\mu(s)$.

b) Express it in terms of the eigenvalues of the appropriate matrices.

2. For the above problem, let

$$\Sigma_0 = \sigma_n^2 I, \quad \Sigma_1 = \Sigma_s + \sigma_n^2 I.$$  

Find $\mu(s)$, $\dot{\mu}(s)$, $\ddot{\mu}(s)$.

3. The reason of using tilted densities and Chernoff bounds is that a straightforward application of the central limit theorem gives misleading results when the region of interests is on the tail of the density. Consider the following example: Let $x_i$ be statistically independent random variables assuming values 0 and 1 with equal probability. We are interested in

$$P[y_N = 1 \sum_{i=1}^{N} x_i \geq 1] = P(A_N).$$

a) Let

$$Z = \frac{y_N - E[y_N]}{\sigma_{y_N}}, \quad \sigma_{y_N}^2 = \text{Var}[y_N]$$

Use a central limit theorem argument to estimate $P(A_N)$. Denote this estimate as $\hat{P}(A_N)$.

b) Calculate $P(A_N)$ exactly.

c) Show that $\frac{\hat{P}(A_N)}{P(A_N)}$ is proportional to $(e^{0.19})^N$, which grows exponentially.

d) Estimate $P(A_N)$ with the Chernoff bound $\exp(\mu_s - s\dot{\mu}(s))$, $s \geq 0$. Denote the estimate as $P_c(A_N)$. Compute $\frac{P_c(A_N)}{P(A_N)}$.

e) Hint: Use the results in Problem 4.

4. Let $x$ be a random variable. Let $h(x)$ be a function of $x$ and

$$h(x) \geq 0, \quad \text{all } x,$$

$$h(x) \geq h(x_0) > 0, \quad \text{all } x \geq x_0.$$
a) Prove that

\[ P(x \geq x_0) \leq \frac{E[h(x)]}{h(x_0)}. \]

b) Let \( h(x) = e^{sx}, \ s \geq 0, \) and \( x_0 = \gamma. \) Use the result in a) to prove that

\[ P(x \geq \gamma) = \int_{\gamma}^{\infty} f(x)dx \leq e^{\mu(s) - s\gamma}, \ s \geq 0. \]

c) Find the \( s \) in b) so that the bound is minimized.